Janet's Algorithm and Systems Theory

Daniel Robertz

Lehrstuhl B für Mathematik

RWTH AACHEN UNIVERSITY

http://wwwb.math.rwth-aachen.de/~daniel

28.11.2012

Outline

- Janet's algorithm
- Module-theoretic approach to linear systems
- Parametrizing linear systems

Janet's algorithm

Janet's algorithm for linear PDEs

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} &= 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} &= 0 \end{cases}$$

find: $u = u(x, y)$ analytic

$$u(x,y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \tfrac{x^2}{2!} + a_{1,1}\, \tfrac{xy}{1!1!} + a_{0,2}\, \tfrac{y^2}{2!} + \ldots$$

Janet's algorithm for linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

find: $u = u(x, y)$ analytic

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm for linear PDEs

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} &= 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} &= 0 \end{cases}$$

find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm for linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm for linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

find: $u = u(x, y)$ analytic

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \frac{x^2}{2!} + a_{1,1}\, \frac{xy}{1!1!} + a_{0,2}\, \frac{y^2}{2!} + \ldots$$

Janet's algorithm computes a vector space basis for power series solutions

(Maurice Janet, $\sim$ 1920)

Janet's algorithm for linear PDEs

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \quad A \\ u_{x,x} - y u_{z,z} & = & 0 & \quad B \end{array}\right.$$

is equivalent to

$$\left\{\begin{array}{rcll} u_{y,y} & = & 0 & \quad A \\ u_{x,x} - y u_{z,z} & = & 0 & \quad B \\ u_{y,z,z} & = & 0 & \quad \frac{1}{2}(\partial_x^2 - y\partial_z^2)A - \frac{1}{2}\partial_y^2 B \\ u_{x,y,y} & = & 0 & \quad \partial_x A \\ u_{z,z,z,z} & = & 0 & \quad \frac{1}{2}(\partial_x^4 - 2y\partial_x^2\partial_z^2 + y^2\partial_z^4)A - \frac{1}{2}(\partial_x^2\partial_y^2 - y\partial_y^2\partial_z^2 + 2\partial_y\partial_z^2)B \\ u_{x,y,z,z} & = & 0 & \quad \frac{1}{2}(\partial_x^3 - y\partial_x\partial_z^2)A - \frac{1}{2}\partial_x\partial_y^2 B \\ u_{x,z,z,z,z} & = & 0 & \quad \frac{1}{2}(\partial_x^5 - 2y\partial_x^3\partial_z^2 + y^2\partial_x\partial_z^4)A - \frac{1}{2}(\partial_x^3\partial_y^2 + y\partial_x\partial_y^2\partial_z^2 - 2\partial_x\partial_y\partial_z^2)B \end{array}\right.$$

Taylor coeff's for $1, z, y, x, z^2, yz, xz, xy, z^3, xz^2, xyz, xz^3$ *arbitrary*,

all other coeff's determined by linear equations

Outline

- Decomposition of multiple closed sets of monomials into disjoint cones
- Janet's algorithm
- (Generalized) Hilbert series, Hilbert polynomial
- Construction of a free resolution
- Janet bases over $\mathbb{Z}$
- Janet bases for Ore algebras

Strategy

$R = K[x_1, \ldots, x_n]$, K field

I ideal of R (or, more generally: M submodule of R^m)

goal: compute "good" generating set for I (Janet basis)

sort terms in a polynomial in a way compatible with multiplication

total ordering $<$ on $\mathbb{M} := \mathrm{Mon}(x_1, \ldots, x_n) := \{x^i \mid i \in (\mathbb{Z}_{\geq 0})^n\}$

use highest terms $\mathrm{lm}(p)$, $\mathrm{lm}(q)$ to decide "divisibility"

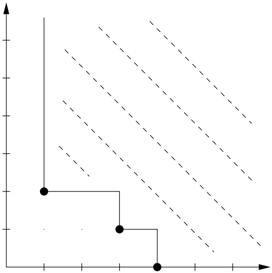
represent $\dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y}$ as $\partial_x \, \partial_y - \partial_y$ in $\mathbb{Q}[\partial_x, \partial_y]$

Multiple closed sets of monomials

$$\mathbb{M} := \mathrm{Mon}(x_1, \ldots, x_n) := \{x^i \mid i \in (\mathbb{Z}_{\geq 0})^n\}$$

$S \subseteq \mathbb{M}$ is $\mathbb{M}$-*multiple closed* if

$$m\, s \in S \quad \forall\, m \in \mathbb{M},\ s \in S$$



$\mathbb{M}$-multiple closed set

generated by $x_1 x_2^2$, $x_1^3 x_2$, x_1^4

$=: \langle\, x_1 x_2^2,\, x_1^3 x_2,\, x_1^4\, \rangle_{\mathbb{M}}$

Multiple closed sets of monomials

Lemma

Every $\mathbb{M}$-multiple closed set $S \subseteq \mathbb{M}$ has a finite generating set.

Proof.

Every seq. $F: m_1, m_2, m_3 \ldots \in \mathbb{M}$ s.t. $m_i \nmid m_j \quad \forall\, i < j$ is finite.

Induction: $n = 1$: clear.

$n \to n+1$: Let $m_1 = x_1^{a_1} \cdots x_n^{a_n}$.

Define subsequence $F^{(j,d)}: \quad m_i = x_1^{b_1} \cdots x_j^{d} \cdots x_n^{b_n}$

We have: $$\bigcup_{1 \le j \le n} \; \bigcup_{0 \le d \le a_j} \{F^{(j,d)}\} = \{F\}$$

By induction, the $\{F^{(j,d)}\}$ are finite.

Multiple closed sets of monomials

Lemma

Every $\mathbb{M}$-multiple closed set $S \subseteq \mathbb{M}$ has a finite generating set.

Cor.

Every ascending sequence of $\mathbb{M}$-multiple closed sets becomes stationary.

Given a finite generating set $\{p_1, \ldots, p_r\}$ for $I \trianglelefteq K[x_1, \ldots, x_n]$,

Janet's algorithm computes

$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k = \mathrm{lm}(I)$ (all $\mathbb{M}$-multiple closed)

where S_0 is generated by $\mathrm{lm}(p_1)$, ..., $\mathrm{lm}(p_r)$

$\Rightarrow$ termination

Decomposition into disjoint cones

Def.

Let $C \subseteq \mathrm{Mon}(x_1, \ldots, x_n)$, $\mu \subseteq \{x_1, \ldots, x_n\}$.

(C, μ) is a *cone* if $\exists\, v \in C$ s.t. $C = \mathrm{Mon}(\mu)\, v$

Variables in μ: *multiplicative variables* (for v)

Variables in $\{x_1, \ldots, x_n\} - \mu$: *non-multiplicative variables* (for v)

Dimension of (C, μ): $|\mu|$

Decomposition into disjoint cones

Def.

Let $S \subseteq \mathbb{M} = \mathrm{Mon}(x_1, \ldots, x_n)$.

$\{ (C_1, \mu_1), \ldots, (C_r, \mu_r) \} \subset \mathcal{P}(\mathbb{M}) \times \mathcal{P}(\{x_1, \ldots, x_n\})$

is a *decomposition of S into disjoint cones* if

each (C_i, μ_i) is a cone and $S = \dot{\bigcup}_{i=1}^{r} C_i$.

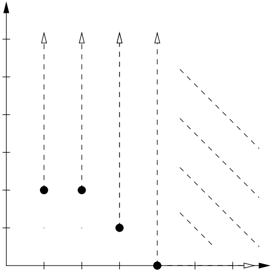
$\{ (v_1, \mu_1), \ldots, (v_r, \mu_r) \}$ is a *decomp. of S into disj. cones*

if $\{ (\mathrm{Mon}(\mu_1)\, v_1, \mu_1), \ldots, (\mathrm{Mon}(\mu_r)\, v_r, \mu_r) \}$ is one.

Decomposition into disjoint cones

Strategy of Janet's algorithm:

Decompose $\mathbb{M}$-multiple closed sets S into disjoint cones.



$$S = \langle\, x_1 x_2^2,\, x_1^3 x_2,\, x_1^4 \,\rangle_{\mathbb{M}}$$

decomposition:

$$\{\, (x_1 x_2^2, \{x_2\}),\, (x_1^2 x_2^2, \{x_2\}),\, (x_1^3 x_2, \{x_2\}),\, (x_1^4, \{x_1, x_2\}) \,\}$$

This can also be done for $\mathrm{Mon}(x_1, \ldots, x_n) - S$.

Janet division

The possible ways of decomposing $\mathbb{M}$-multiple closed sets

into disjoint cones are studied as

involutive divisions (Gerdt, Blinkov et. al.)

Janet division:

Let $G \subset \mathbb{M} = \mathrm{Mon}(x_1, \ldots, x_n)$ be finite.

For a cone with vertex $v = x_1^{a_1} \cdots x_n^{a_n} \in G$

x_i is a *multiplicative variable* iff

$$a_i = \max\{\, b_i \mid x^b \in G;\, b_j = a_j \;\forall\, j < i \,\}.$$

Janet division

For $v = x_1^{a_1} \cdots x_n^{a_n}$:

$$x_i \in \mu \iff a_i = \max\{ b_i \mid x^b \in G;\, b_j = a_j \;\forall\, j < i \}.$$

Example: $G = \{ y\,z,\ x\,y\,z,\ x^2\,y\,z,\ x^2\,y^2 \}$

$y\,z$	
$x\;\;y\,z$	
$x^2\,y\,z$	
$x^2\,y^2$	

Janet division

For $v = x_1^{a_1} \cdots x_n^{a_n}$:

$$x_i \in \mu \iff a_i = \max\{\, b_i \mid x^b \in G;\, b_j = a_j \ \forall\, j < i \,\}.$$

Example: $G = \{\, y\,z,\ x\,y\,z,\ x^2\,y\,z,\ x^2\,y^2 \,\}$

$y\,z$	$*$	y	z
$x\ \ y\,z$	$*$	y	z
$x^2\,y\,z$	x	$*$	z
$x^2\,y^2$	x	y	z

Decomposition into disjoint cones

Decompose(G, η) $\qquad G \subset \mathrm{Mon}(x_1, \ldots, x_n), \quad \emptyset \neq \eta \subseteq \{x_1, \ldots, x_n\}$

$G \leftarrow \{g \in G \mid \nexists\, h \in G : h \mid g\}$

if $|G| \leq 1$ or $|\mu| = 1$ then

return $\{(m, \eta) \mid m \in G\}$

else

$y \leftarrow y_a$ with $a = \min\{i \mid 1 \leq i \leq n,\, y_i \in \eta\}$

$d \leftarrow \max\{\deg_y(g) \mid g \in G\}$

$G_i \leftarrow \{g \in G \mid \deg_y(g) = i\}, \qquad i = 0, \ldots, d$

$G_i \leftarrow G_i \cup \bigcup_{j=0}^{i-1} \{y^{i-j} g \mid g \in G_j\}, \qquad i = 1, \ldots, d$

$T_d \leftarrow \{\, (m, \zeta \cup \{y\}) \mid (m, \zeta) \in$ Decompose$(G_d, \eta - \{y\})\,\}$

$T_i \leftarrow$ Decompose$(G_i, \eta - \{y\}), \qquad i = 0, \ldots, d-1$

return $\bigcup_{i=0}^{d} T_i$

fi

Janet reduction

NF(p, T, $\prec$) $\qquad p \in K[x_1, \ldots, x_n], \quad T = \{ (d_1, \mu_1), \ldots, (d_s, \mu_s) \}$

$r \leftarrow 0$

while $p \neq 0$ do

if $\exists (d, \mu) \in T : \mathrm{lm}(p) \in \mathrm{Mon}(\mu)\ \mathrm{lm}(d)$ then

$p \leftarrow p - \frac{\mathrm{lc}(p)}{\mathrm{lc}(d)} \frac{\mathrm{lm}(p)}{\mathrm{lm}(d)} d$

else

$r \leftarrow r + \mathrm{lc}(p)\ \mathrm{lm}(p)$

$p \leftarrow p - \mathrm{lc}(p)\ \mathrm{lm}(p)$

fi

od

return r

Disj. cones $\Rightarrow$ course of Alg. is uniquely determined

Janet's algorithm

JanetBasis(F, $\prec$) $\qquad$ $F \subseteq K[x_1, \ldots, x_n]$ finite

$G \leftarrow F$

do

$\quad$ $G \leftarrow$ auto-reduce G

$\quad$ $J \leftarrow \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ s.t. $\{ (\mathrm{lm}(p_1), \mu_1), \ldots, (\mathrm{lm}(p_r), \mu_r) \}$

$\qquad$ decomp. into disj. cones of $\langle \mathrm{lm}(G) \rangle_{\mathbb{M}}$

$\quad$ $P \leftarrow \{ \mathrm{NF}(x \cdot p, J) \mid (p, \mu) \in J,\ x \notin \mu \}$

$\quad$ $G \leftarrow \{ p \mid (p, \mu) \in J \} \cup P$

while $\quad P \neq \{0\}$

return J

Janet basis

Janet basis $J = \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for $I = \langle F \rangle$

- Invant of the loop:

  G (or $\{p_1, \ldots, p_r\}$) always forms a gen. set for I

- $\displaystyle\bigcup_{i=1}^{r} \mathrm{Mon}(\mu_i) p_i$ is a K-basis of I.

  Linear independence: clear.

  $p \in I$: $\quad p = \displaystyle\sum_{i=1}^{r} c_i \, p_i$

Example

Let $I := \langle\, g_1,\, g_2 \,\rangle \trianglelefteq K[x, y]$, $\qquad g_1 := x^2 - y, \quad g_2 := xy - y$.

Let $>$ be degrevlex, $\; x > y$.

Decomposition into disjoint cones of $\quad \langle\, \mathrm{lm}(g_1),\, \mathrm{lm}(g_2) \,\rangle$:

$\{\, (x^2,\, \{x, y\}),\, (xy, \{y\}) \,\}$

$$f := x \cdot g_2 = x^2 y - xy \in I, \qquad f = \sum_{i=1}^{2} c_i\, g_i?$$

Reduction of f modulo g_1, g_2 yields: $\qquad g_3 := y^2 - y \in I$

$\{\, (g_1, \{x, y\}),\, (g_2, \{y\}),\, (g_3, \{y\}) \,\}$ $\qquad$ (minimal) Janet basis for I

(Generalized) Hilbert series

Janet basis $J = \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for I

We have $\operatorname{lm}(I) = \langle \operatorname{lm}(p_1), \ldots, \operatorname{lm}(p_r) \rangle_{\mathbb{M}}$.

Generalized Hilbert series

$$H_I(x_1, \ldots, x_n) = \sum_{i=1}^{r} \operatorname{lm}(p_i) \prod_{x_j \in \mu_i} \frac{1}{1 - x_j}$$

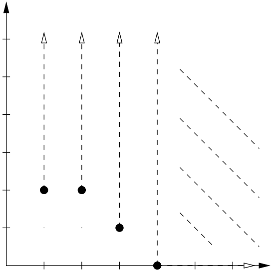
enumerates a K-basis of $\operatorname{lm}(I)$.

$H_I(t, \ldots, t)$ is the usual Hilbert series.

Example

Janet basis for I

$$J = \{\, (x_1 x_2^2,\, \{x_2\}),\, (x_1^2 x_2^2,\, \{x_2\}),\, (x_1^3 x_2,\, \{x_2\}),\, (x_1^4,\, \{x_1, x_2\})\,\}$$



generalized Hilbert series:

$$\frac{x_1 x_2^2}{1 - x_2} + \frac{x_1^2 x_2^2}{1 - x_2} + \frac{x_1^3 x_2}{1 - x_2} + \frac{x_1^4}{(1 - x_1)(1 - x_2)}$$

Hilbert polynomial

Janet basis of M: $\{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$

$$\begin{aligned} H_M(t, \ldots, t) &= \sum_{k \geq 0} \dim_K M_k \, t^k \\ &= \sum_{i=1}^{r} t^{\deg(p_i)} \frac{1}{(1-t)^{|\mu_i|}} \\ &= \sum_{i=1}^{r} t^{\deg(p_i)} \sum_{j \geq 0} \binom{|\mu_i| + j - 1}{j} t^j \end{aligned}$$

Coeff. of t^k in $H_M(t, \ldots, t)$? For $k \geq \max\{\deg(p_i) \mid i = 1, \ldots, r\}$:

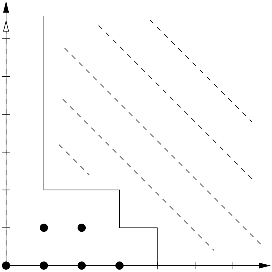
$$\dim_K M_k = \sum_{i=1}^{r} \binom{|\mu_i| + k - \deg(p_i) - 1}{k - \deg(p_i)}$$

Example

$S = \text{lm}(I)$

Decomp. of $\text{Mon}(x_1, \ldots, x_n) - S$ into disjoint cones

$\rightsquigarrow$ generalized Hilbert series enum. a K-basis of $K[x_1, \ldots, x_n]/I$



generalized Hilbert series:

$$\frac{1}{1 - x_2} + x_1 + x_1 x_2 + x_1^2 + x_1^2 x_2 + x_1^3$$

Hilbert polynomial:

$$\sum_{i=1}^{6} \binom{|\mu_i| + k - \deg(p_i) - 1}{k - \deg(p_i)} = 1$$

Free resolution

Janet basis $J = \{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for I

We have $x_j \, p_i = \sum_k \alpha_{i,j,k} \, p_k, \qquad x_j \notin \mu_i, \quad \alpha_{i,j,k} \in K[\mu_k]$

Define $\pi : R^{|J|} \to R : \hat{p}_i \mapsto p_i$. ($\hat{p}_i$ std. gen.)

Prop.

$$x_j \, \hat{p}_i - \sum_k \alpha_{i,j,k} \, \hat{p}_k, \qquad x_j \notin \mu_i, \quad i = 1, \ldots, r,$$

form a Janet basis of $\ker \pi$ for a suitable monomial ordering.

$\rightsquigarrow$ construction of a free resolution of R/I.

Example

$$R = K[x_1, x_2, x_3], \qquad x_1 > x_2 > x_3, \qquad I = (x_1, x_2, x_3)$$

Janet basis:

$$\begin{array}{c|ccc} x_1 & x_1 & x_2 & x_3 \\ x_2 & * & x_2 & x_3 \\ x_3 & * & * & x_3 \end{array}$$

normal form computation:

$$\begin{array}{l} x_1 \cdot x_2 - x_2 \cdot x_1 \\ x_1 \cdot x_3 - x_3 \cdot x_1 \\ x_2 \cdot x_3 - x_3 \cdot x_2 \end{array}$$

$$R^{1\times 3} \xrightarrow{\begin{pmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix}} R^{1\times 3} \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

Example

$$R = K[x_1, x_2, x_3], \qquad x_1 > x_2 > x_3, \qquad I = (x_1, x_2, x_3)$$

Janet basis:

$$\begin{array}{ccc|ccc} [-x_2 & \underline{x_1} & 0] & x_1 & x_2 & x_3 \\ [-x_3 & 0 & \underline{x_1}] & x_1 & x_2 & x_3 \\ [\;0 & -x_3 & \underline{x_2}] & * & x_2 & x_3 \end{array}$$

normal form computation:

$$x_1 \cdot [\quad 0 \;\; -x_3 \quad x_2] - x_2 \cdot [-x_3 \quad 0 \quad x_1] + x_3 \cdot [-x_2 \quad x_1 \quad 0]$$

$$0 \to R \xrightarrow{\begin{pmatrix} x_3 & -x_2 & x_1 \end{pmatrix}} R^{1\times 3} \xrightarrow{\begin{pmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix}} R^{1\times 3} \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} R \to R/I \to 0$$

Free resolution

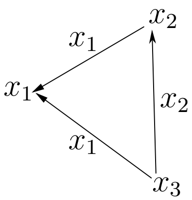
Prop.

$$x_j \, \hat{p}_i - \sum_k \alpha_{i,j,k} \, \hat{p}_k, \qquad x_j \notin \mu_i, \quad i = 1, \ldots, r,$$

form a Janet basis of $\ker \pi$ w.r.t. $\prec$.

Choose a total order $\ll$ on J s.t.

$p_k \ll p_l$ if $\exists$ path from p_l to p_k in the Janet graph.



$$x^i \, \hat{p}_k \prec x^j \, \hat{p}_l \quad :\Longleftrightarrow \quad \begin{cases} & x^i \, \mathrm{lm}(p_k) < x^j \, \mathrm{lm}(p_l) \\ \text{or} & x^i \, \mathrm{lm}(p_k) = x^j \, \mathrm{lm}(p_l) \quad \text{and} \quad p_k \ll p_l \end{cases}$$

Consequences

$\{\,(p_1, \mu_1),\, \ldots,\, (p_r, \mu_r)\,\}$ Janet basis for $I \trianglelefteq R$

- can decide ideal membership
- normal form for residue classes modulo I
- enumeration of a K-basis of I and a K-basis of R/I
  (generalized Hilbert series)
- can easily determine Hilbert polynomial
- can read off a free resolution of R/I
- every Janet basis is a Gröbner basis

Janet bases over $\mathbb{Z}$

<u>NF(p, T, $\prec$)</u> $\qquad p \in \mathbb{Z}[x_1, \ldots, x_n], \qquad T = \{ (d_1, \mu_1), \ldots, (d_l, \mu_l) \}$

$r \leftarrow 0$

while $p \neq 0$ do

  if $\exists \, (d, \mu) \in T : \mathrm{lm}(p) \in \mathrm{Mon}(\mu) \, d$ then

    write $\mathrm{lc}(p) = a \cdot \mathrm{lc}(d) + b, \qquad |b| < |\,\mathrm{lc}(d)|$

    if $a \neq 0$ then

      $p \leftarrow p - a \, \frac{\mathrm{lm}(p)}{\mathrm{lm}(d)} \, d$

    else

      move leading term from p *to* r

    fi

  else

    move leading term from p *to* r

  fi

od

return r

Janet bases for Ore algebras

Skew polynomial ring $A[\partial; \sigma, \delta]$:

A domain and K-algebra

$\sigma : A \to A$ K-algebra endomorphism

$\delta : A \to A$ σ-derivation, i.e.

$$\delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b, \qquad a, b \in A$$

$$A[\partial; \sigma, \delta] = \left\{ \sum_{\text{fin.}} a_i\,\partial^i \mid a_i \in A,\ i \in \mathbb{Z}_{\geq 0} \right\}$$

with commutation rule

$$\partial\,a = \sigma(a)\,\partial + \delta(a), \qquad a \in A$$

Janet bases for Ore algebras

Ore algebra $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$:

$A = K$ or $A = K[x_1, \ldots, x_n]$

$\sigma_i : D \to D$ K-algebra endomorphisms

$\delta_i : D \to D$ σ_i-derivations

$$A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m] = \left\{ \sum_{\text{fin.}} a_i \, \partial^i \mid a_i \in A, \, i \in (\mathbb{Z}_{\geq 0})^m \right\}$$

with commutation rules

$$\begin{aligned} \partial_i \, a &= \sigma_i(a) \, \partial_i + \delta_i(a), \qquad a \in A, \\ \partial_i \, \partial_j &= \partial_j \, \partial_i \end{aligned}$$

Janet bases for Ore algebras

- Weyl algebra: *ordinary differential equations*

  $A_1 = K[t][\frac{d}{dt}]$ $\boxed{\frac{d}{dt}\,a = a\,\frac{d}{dt} + \frac{da}{dt}}$

- Weyl algebra: *partial differential equations*

  $A_n = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$ $\boxed{\partial_i\,x_j = x_j\,\partial_i + \delta_{ij}}$

- $B_n = K(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$

- Shift operators: *difference equations*

  $S_h = K[t][\delta_h]$ $\boxed{\delta_h\,t = (t - h)\,\delta_h}$

- combinations . . .

Janet bases for Ore algebras

$D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_m]$

I *left* ideal of D generated by p_1, ..., p_r

- *normal form* for elements of D:

  use $\partial_i \, x_j = \sigma_i(x_j) \, \partial_i + \ldots$

  to move all ∂_i to the right of every x_j

- $\mathbb{M} := \{ \, x^i \, \partial^j \mid i \in (\mathbb{Z}_{\geq 0})^n, \, j \in (\mathbb{Z}_{\geq 0})^m \, \}$

  consider $\mathbb{M}$-multiple closed set generated by

  the normal forms of $\mathrm{lm}(p_i)$, $i = 1, \ldots, r$

- decomp. into disj. cones as before

- reduction: all multiplications from the *left*

Janet bases for Ore algebras

$D = K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_m]$

I left ideal of D generated by p_1, ..., p_r

For *termination* of the algorithm, assume that

$$\partial_i \, x_j = (c_{i,j} \, x_j + d_{i,j}) \, \partial_i + e_{i,j}$$

where $c_{i,j} \in K - \{0\}$, $d_{i,j} \in K$,

$e_{i,j} \in K[x_1, \ldots, x_n]$ with $\deg(e_{i,j}) \leq 1$

Maple packages...

...at Lehrstuhl B für Mathematik, RWTH Aachen University

implementing the involutive basis technique:

`Involutive` / `Janet`

`JanetOre`

`LDA` (**L**inear **D**ifference **A**lgebra)

in cooperation with V. P. Gerdt & Y. A. Blinkov

Involutive

- Janet (-like Gröbner) bases for submodules of free modules over a commutative polynomial ring
- coefficients: rationals or finite fields and field extensions, and rational integers
- Janet division, Janet-like division
- term orderings:
  degrevlex, plex
  TOP / POT
  block / elimination orderings

web: `http://wwwb.math.rwth-aachen.de/Janet`

Involutive

- Analogues of Buchberger's criteria can be selected
- Interface to C++:
  call fast routines when needed or
  switch to fast routines for the whole Maple session
- Syzygies, Hilbert series, etc.
- Applications:
  commutative algebra
  solving systems of algebraic equations

web: `http://wwwb.math.rwth-aachen.de/Janet`

Main procedures of Involutive

`InvolutiveBasis`

compute Janet(-like Gröbner) basis

`PolInvReduce`

involutive reduction modulo Janet basis

`FactorModuleBasis`

vector space basis of residue class module

`Syzygies`

syzygy module

`PolResolution`

free resolution

`PolHilbertSeries`, `PolHilbertPolynomial`, etc.

combinatorial devices

`PolMinPoly`, `PolRepres`, etc.

computing in residue class rings

ginv

- C++ module for Python
- comp. of Gröbner bases using involutive algorithms
- polynomials, differential / difference equations
- open source software
- originated by V. P. Gerdt, Y. A. Blinkov
- contributions by LBfM
- coefficients: rationals or finite fields and some algebraic and transcendental field extensions
- term orderings: degrevlex (TOP / POT), lex, product orderings
- see web page for timings

web: `http://invo.jinr.ru`

ginv

```
import ginv
st = ginv.SystemType("Polynomial")
im = ginv.MonomInterface("DegRevLex", st, ['x', 'y'])
ic = ginv.CoeffInterface("GmpZ", st)
ip = ginv.PolyInterface("PolyList", st, im, ic)
iw = ginv.WrapInterface("CritPartially", ip)
iD = ginv.DivisionInterface("Janet", iw)
eqs = ["x^2+y^2", ...]
basis = ginv.basisBuild("TQ", iD, eqs)
```

Module-theoretic approach to linear systems

Linear Systems

D ring (field, integral domain, Ore algebra, ...)

$R \in D^{q \times p}$, $\mathcal{F}$ left D-module

$$R\,y = 0, \qquad y \in \mathcal{F}^p.$$

"optimal" answer for us: $P \in D^{p \times r}$ s.t. $\ker(R.) = \mathrm{im}(P.)$

not possible in general

Example. $D = k$ a (skew) field.

- Gaussian elimination singles out parameters
- injective parametrization

Example

de Rham complex:

$D = \mathbb{R}[\partial_x, \partial_y, \partial_z]$, e.g. $\mathcal{F} = C^\infty(\Omega)$, $\Omega \subseteq \mathbb{R}^3$ convex

$$0 \to \mathbb{R} \to \mathcal{F} \xrightarrow{\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} .} \mathcal{F}^{3\times 1} \xrightarrow{\begin{pmatrix} 0 & \partial_z & -\partial_y \\ -\partial_z & 0 & \partial_x \\ \partial_y & -\partial_x & 0 \end{pmatrix} .} \mathcal{F}^{3\times 1} \xrightarrow{\begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} .} \mathcal{F} \to 0$$

$$0 \leftarrow M \leftarrow D \xleftarrow{.\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}} D^{1\times 3} \xleftarrow{.\begin{pmatrix} 0 & \partial_z & -\partial_y \\ -\partial_z & 0 & \partial_x \\ \partial_y & -\partial_x & 0 \end{pmatrix}} D^{1\times 3} \xleftarrow{.\begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix}} D \leftarrow 0$$

Module-theoretic approach to linear systems

$\Sigma : \quad R\,y = 0, \qquad R \in D^{q\times p}$

$D =$ ring of functional operators

$M = D^{1\times p}/D^{1\times q}\,R$ independent of eq.'s chosen for Σ

$\mathcal{F} =$ signal space

If $\mathcal{F}$ is an injective cogenerator for ${}_D\mathcal{M}$ then

$$M \xrightarrow{\hom_D(\,\cdot\,,\mathcal{F})} \mathrm{Sol}_{\mathcal{F}}(M)$$

is a categorical duality.

Malgrange, Sato, Kashiwara, Oberst, Fliess, Mounier, Pommaret, Quadrat, Willems, Zerz, ...

Module-theoretic approach to linear systems

$\Sigma: \quad R\,y = 0, \qquad R \in D^{q\times p}, \qquad M = D^{1\times p}/D^{1\times q}\,R$

$\mathcal{F}$ injective cogenerator for ${}_D\mathcal{M}$:

$$0 \longleftarrow M \longleftarrow D^{1\times p} \xleftarrow{.R} D^{1\times q} \xleftarrow{.S} D^{1\times r} \qquad \text{exact}$$

if and only if (apply $\hom_D(\,\cdot\,,\mathcal{F})$)

$$0 \longrightarrow \mathrm{Sol}_{\mathcal{F}}(M) \longrightarrow \mathcal{F}^p \xrightarrow{R.} \mathcal{F}^q \xrightarrow{S.} \mathcal{F}^r \qquad \text{exact.}$$

Fundamental principle (Ehrenpreis, Malgrange, Palamodov)

for $D = \mathbb{C}[\partial_1, \ldots, \partial_n]$ acting by differentiation on $\mathcal{F}$:

e.g. $\mathcal{F} \in \{$ complex-valued C^∞-functions on $\mathbb{R}^n$,

complex-valued distributions on $\mathbb{R}^n$, formal / convergent power series $\}$

Injective Cogenerator

$$R_1\, u = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad R_1 := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \in D^{2\times 1}, \qquad D = K[\partial_x, \partial_y]$$

compatibility condition:

$$R_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \qquad R_2 := (\partial_y \quad -\partial_x) \in D^{1\times 2},$$

$$0 \longleftarrow M \longleftarrow D \xleftarrow{.R_1} D^{1\times 2} \xleftarrow{.R_2} D \longleftarrow 0 \qquad \text{exact}$$

$$0 \longrightarrow \mathrm{Sol}_{\mathcal{F}}(M) \longrightarrow \mathcal{F} \xrightarrow{(R_1).} \mathcal{F}^{2\times 1} \xrightarrow{(R_2).} \mathcal{F} \longrightarrow 0 \qquad \text{exact}$$

Parametrizing linear systems

Parametrization and torsion-freeness

Let M be given by the finite presentation

$$0 \longleftarrow M \xleftarrow{\rho} D^{1\times p} \xleftarrow{.R} D^{1\times q}.$$

Assume P is a parametrization, i.e.

$$D^{1\times r} \xleftarrow{.P} D^{1\times p} \xleftarrow{.R} D^{1\times q}$$

is an exact sequence of left D-modules. Then M is torsion-free:

$$\begin{array}{c} \mathcal{F}^r \\ \downarrow \iota^* \\ \mathrm{Sol}_{\mathcal{F}}(M) \\ \downarrow \\ 0 \end{array} \qquad \begin{array}{ccccc} & D^{1\times r} & \xleftarrow{.P} & D^{1\times p} & \xleftarrow{.R} & D^{1\times q} \\ & \uparrow \iota & \swarrow \rho & & & \\ & M & & & & \\ \swarrow & \uparrow & & & & \\ 0 & 0 & & & & \end{array}$$

Example

F. Dubois, N. Petit, and P. Rouchon,

Motion Planning and Nonlinear Simulations for a Tank Containing a Fluid,

Proc. ECC, Karlsruhe (Germany), 1999.

$$\left\{ \begin{array}{lcl} \phi_1(t-2) + \phi_2(t) - 2\,\dot{\phi}_3(t-1) & = & 0, \\ \phi_1(t) + \phi_2(t-2) - 2\,\dot{\phi}_3(t-1) & = & 0. \end{array} \right.$$

$D := \mathbb{Q}\,[\partial, \delta]$ (differential time-delay operators)

$$R := \left(\begin{array}{ccc} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{array} \right) \in D^{2\times 3},$$

$M := D^{1\times 3}/D^{1\times 2}\,R$

Parametrizability test

$$M = D^{1\times p}/D^{1\times q}\,R \qquad \rightsquigarrow \qquad M^{\top} = D^{q\times 1}/R\,D^{p\times 1}$$

$$D^{1\times 1} \xrightarrow{\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ \delta^2+1 \end{pmatrix}.} D^{3\times 1} \xrightarrow{\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{pmatrix}.} D^{2\times 1} \longrightarrow M^{\top} \longrightarrow 0$$

$$D^{1\times 1} \xleftarrow{.\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ \delta^2+1 \end{pmatrix}} D^{1\times 3} \xleftarrow{.\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{pmatrix}} D^{1\times 2} \qquad \text{not exact}$$

$$D^{1\times 1} \xleftarrow{.\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ \delta^2+1 \end{pmatrix}} D^{1\times 3} \xleftarrow{.\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{pmatrix}} D^{1\times 2}$$

$$D^{1\times 3} \longleftarrow D^{1\times 2}$$

$$R' := \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\delta^2-1 & 2\,\delta\,\partial \end{pmatrix}$$

Parametrizability test

$$D^{1\times 1} \xleftarrow{.\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ \delta^2+1 \end{pmatrix}} D^{1\times 3} \xleftarrow{.\begin{pmatrix} \delta^2 & 1 & -2\,\delta\,\partial \\ 1 & \delta^2 & -2\,\delta\,\partial \end{pmatrix}} D^{1\times 2}$$

$$D^{1\times 3} \longleftarrow D^{1\times 2}, \quad R' := \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\delta^2-1 & 2\,\delta\,\partial \end{pmatrix}$$

We have $\quad t(M) = D^{1\times 2}\, R' / D^{1\times 2}\, R \neq 0.$

In particular, $\quad (\delta^2-1)\,(\phi_1(t)-\phi_2(t)) = 0, \qquad \phi_1-\phi_2$ is 2-periodic.

$\begin{pmatrix} 2\,\delta\,\partial \\ 2\,\delta\,\partial \\ \delta^2+1 \end{pmatrix}$ is a parametrization of the subsystem

$$\begin{cases} \phi_1(t)-\phi_2(t) = 0, \\ -\phi_2(t-2)-\phi_2(t)+2\,\dot{\phi}_3(t-1) = 0. \end{cases}$$

Parametrizability test

In fact, we compute $\quad \mathrm{ext}^1_D(M^\top, D) \cong t(M)$.

$$
\begin{array}{ccccccccc}
 & 0 & & & & & & & \\
 & \downarrow & & & & & & & \\
 & t(M) & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longleftarrow & M & \longleftarrow & F_0 & \longleftarrow & F_1 & \longleftarrow & \hom_D(M^\top, D) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longleftarrow & K \otimes_D M & \longleftarrow & K \otimes_D F_0 & \longleftarrow & K \otimes_D F_1 & \longleftarrow & \hom_D(M^\top, K) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longleftarrow & (K/D) \otimes_D M & \longleftarrow & (K/D) \otimes_D F_0 & \longleftarrow & (K/D) \otimes_D F_1 & \longleftarrow & \hom_D(M^\top, K/D) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & \mathrm{ext}^1_D(M^\top, D) & \\
 & & & & & & & \downarrow & \\
 & & & & & & & 0 &
\end{array}
$$

Parametrizability test

We can compute

$$R' := \begin{pmatrix} 1 & -1 & 0 \\ 0 & -\delta^2 - 1 & 2\,\partial\,\delta \end{pmatrix}, \qquad R'' := \begin{pmatrix} \delta^2 & -1 \\ 1 & -1 \end{pmatrix}$$

which satisfy $R = R''\,R'$. Here we have $\ker(R') = 0$.

$\Rightarrow \qquad t(M) \cong D^{1\times 2}/(D^{1\times 2}\,R''), \qquad M/t(M) \cong D^{1\times 3}/(D^{1\times 2}\,R')$.

$M/t(M)$ corresponds to the parametrizable subsystem

$$\begin{cases} \phi_1(t) - \phi_2(t) = 0, \\ -\phi_2(t-2) - \phi_2(t) + 2\,\dot{\phi}_3(t-1) = 0. \end{cases}$$

System	**Module**	**Homological Algebra**
autonomous elements	$t(M) \neq 0$	$\mathrm{ext}^1_D(M^T, D) \neq 0$
controllable, parametrizable	$t(M) = 0$	$\mathrm{ext}^1_D(M^T, D) = 0$
parametrization is parametrizable	reflexive	$\mathrm{ext}^i_D(M^T, D) = 0$, $i = 1, 2$
...	...	...
...	projective	$\mathrm{ext}^i_D(M^T, D) = 0$, $1 \leq i \leq \mathrm{gld}(D)$
flatness	free	...

Contributions to this classification: Pommaret-Quadrat, Oberst, Fliess, Mounier, ...

References

M. Janet,
Leçons sur les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, Paris, 1929

C. Méray,
Démonstration générale de l'existence des intégrales des équations aux dérivées partielles,
J. de mathématiques pures et appliquées, 3e série, tome VI, 1880

C. Riquier,
Les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, Paris, 1910

J. F. Ritt,
Differential Algebra,
Dover, 1966

References

W. Plesken, D. Robertz,
Janet's approach to presentations and resolutions for polynomials and linear pdes,
Archiv der Mathematik, 84 (1), 2005, pp. 22–37

Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz,
The Maple Package "Janet": I. Polynomial Systems and *II. Linear Partial Differential Equations*,
Proceedings of CASC 2003, pp. 31–40 resp. pp. 41–54

F.-O. Schreyer,
Die Berechnung von Syzygien mit dem verallgemeinerten Weierstraßschen Divisionssatz und eine Anwendung auf analytische Cohen-Macaulay-Stellenalgebren minimaler Multiplizität,
Diploma Thesis, Univ. Hamburg, Germany, 1980

References

D. Robertz,
Formal Computational Methods for Control Theory,
PhD thesis, RWTH Aachen University, 2006, available at
`http://darwin.bth.rwth-aachen.de/opus/volltexte/2006/1586`

D. Robertz,
Janet bases and applications,
in: M. Rosenkranz, D. Wang, *Gröbner Bases in Symbolic Analysis*,
Radon Series Comp. Appl. Math., de Gruyter, 2007

D. Robertz,
Noether normalization guided by monomial cone decompositions,
J. of Symbolic Computation, 44 (10), 2009, pp. 1359–1373

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Habilitationsschrift, accepted by the Faculty of Mathematics,
Computer Science and Natural Sciences, RWTH Aachen University, 2012

References

W. Plesken, D. Robertz,
Constructing Invariants for Finite Groups,
Experimental Mathematics, 14 (2), 2005, pp. 175–188

W. Plesken, D. Robertz,
Representations, commutative algebra, and Hurwitz groups,
J. Algebra, 300 (2006), 2006, pp. 223–247

W. Plesken, D. Robertz,
Elimination for coefficients of special characteristic polynomials,
Experimental Mathematics 17 (4), 2008, pp. 499–510

W. Plesken, D. Robertz,
Linear Differential Elimination for Analytic Functions,
Mathematics in Computer Science, 4 (2–3), 2010, pp. 231–242

References

V. P. Gerdt, Y. A. Blinkov,
Involutive bases of polynomial ideals. Minimal involutive bases,
Mathematics and Computers in Simulation, 45, 1998

Y. A. Blinkov, V. P. Gerdt, D. A. Yanovich,
Construction of Janet Bases, I. Monomial Bases, II. Polynomial Bases,
Proceedings of CASC 2001

V. P. Gerdt,
Involutive Algorithms for Computing Gröbner Bases,
Proc. "Computational commutative and non-commutative algebraic geometry" (Chishinau, June 6-11, 2004), IOS Press, 2005

V. P. Gerdt, Y. A. Blinkov, V. V. Mozzhilkin,
Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations,
Symmetry, Integrability and Geometry: Methods and Applications, 2006

References

V. P. Gerdt, D. Robertz,
A Maple Package for Computing Gröbner Bases for Linear Recurrence Relations,
Nuclear Instruments and Methods in Physics Research A, 559 (1), 2006, pp. 215–219

V. P. Gerdt, D. Robertz,
Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification,
in: S. M. Watt (ed.), Proceedings of ISSAC 2010, TU München, Germany, pp. 53–59

V. P. Gerdt, D. Robertz,
Computation of Difference Gröbner Bases,
Computer Science Journal of Moldova, 20 (2), 2012, pp. 203–226

References

F. Chyzak, B. Salvy,
Non-commutative elimination in Ore algebras proves multivariate identities,
J. Symbolic Computation, 26, 1998

V. Levandovskyy,
Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation,
PhD thesis, Univ. Kaiserslautern, Germany, 2005

V. P. Gerdt, D. A. Yanovich,
Experimental Analysis of Involutive Criteria,
"Algorithmic Algebra and Logic 2005", April 3-6, 2005, Passau, Germany

J. Apel, R. Hemmecke,
Detecting unnecessary reductions in an involutive basis computation,
J. Symbolic Computation, 40, 2005

References

W. W. Adams, P. Loustaunau,
An Introduction to Gröbner Bases,
AMS, 1994

T. Becker and V. Weispfenning,
Gröbner Bases. A Computational Approach to Commutative Algebra,
Springer, 1993

D. Cox, J. Little, D. O'Shea,
Ideals, Varieties, and Algorithms,
Springer, 1992

D. Eisenbud,
Commutative Algebra with a View Toward Algebraic Geometry,
Springer, 1995

References

B. Malgrange,
Systèmes à coefficients constants,
Séminaire Bourbaki 246:79–89, 1962–63.

U. Oberst,
Multidimensional constant linear systems,
Acta Appl. Math. 20:1–175, 1990.

J.-F. Pommaret and A. Quadrat,
Algebraic analysis of linear multidimensional control systems,
IMA Journal of Control and Information 16 (3):275–297, 1999.

J.-F. Pommaret and A. Quadrat,
A functorial approach to the behavior of multidimensional control systems, Applied Mathematics and Computer Science, 13:7–13, 2003.

J.-F. Pommaret
Partial Differential Control Theory
Kluwer, 2001

References

R. Baer,
Erweiterungen von Gruppen und ihren Isomorphismen,
Math. Z. 38 (1):375–416, 1934.

H. Cartan and S. Eilenberg,
Homological Algebra,
Princeton University Press, 1956.

S. MacLane,
Homology,
Springer, 1995.

J. J. Rotman,
An Introduction to Homological Algebra,
Academic Press, 1979.

References

F. Chyzak and A. Quadrat and D. Robertz,
Effective algorithms for parametrizing linear control systems over Ore algebras, Applicable Algebra in Engineering, Communications and Computing 16 (5):319–376, 2005.

A. Quadrat and D. Robertz,
Parametrizing all solutions of uncontrollable multidimensional linear systems, in: 16th IFAC World Congress, Prague, 2005.

A. Quadrat and D. Robertz,
On the blowing-up of stably free behaviours,
44th IEEE CDC and ECC, Seville, 2005.

References

Quadrat, A. and Robertz, D.,
On the Monge problem and multidimensional optimal control,
in: 17th MTNS, Kyoto, Japan, 2006.

Quadrat, A. and Robertz, D.,
Baer's extension problem for multidimensional linear systems,
in: 18th MTNS, Virginia Tech, USA, 2008.

Quadrat, A. and Robertz, D.,
Computation of bases of free modules over the Weyl algebras,
J. Symbolic Computation, 42:1113–1141, 2007

Quadrat, A. and Robertz, D.,
Controllability and differential flatness of linear analytic ordinary differential systems, in: 19th MTNS, Budapest, 2010.

References

A. Quadrat,
Systèmes et Structures: Une approche de la théorie mathématique des systèmes par l'analyse algébrique constructive,
Habilitation thesis, Univ. de Nice Sophia Antipolis, 2010.

T. Cluzeau and A. Quadrat,
Factoring and decomposing a class of linear functional systems, Linear Algebra Appl. 428(1):324–381, 2008.

A. Fabiańska and A. Quadrat,
Applications of the Quillen-Suslin theorem to multidimensional systems theory, in: Gröbner bases in control theory and signal processing, pp. 23–106, Radon Ser. Comput. Appl. Math. 3, de Gruyter, 2007.

References

M. Barakat and D. Robertz,
homalg: A meta-package for homologial algebra,
Journal of Algebra and Its Applications 7 (3):299–317, 2008.

F. Chyzak and A. Quadrat and D. Robertz,
OreModules: A symbolic package for the study of multidimensional linear systems, in: Chiasson, J. and Loiseau, J.-J. (eds.), *Applications of Time-Delay Systems*, LNCIS 352, 233–264, Springer, 2007.

T. Cluzeau and A. Quadrat,
OreMorphisms: A homological algebra package for factoring and decomposing linear functional systems,
in: Loiseau, J.-J., Michiels, W., Niculescu, S.-I., Sipahi, R. (eds.),
Topics in Time-Delay Systems: Analysis, Algorithms and Control,
LNCIS, Springer, 2008.

References

M. Fliess,
Some basic structural properties of generalized linear systems,
Systems & Control Letters 15:391–396, 1990.

M. Fliess, J. Lévine, P. Martin, and P. Rouchon,
Flatness and defect of nonlinear systems: introductory theory and examples,
Int. J. Control 61:1327–1361, 1995.

J. W. Polderman and J. C. Willems,
Introduction to Mathematical Systems Theory: A Behavioral Approach,
TAM 26, Springer, 1998.

H. Pillai and S. Shankar,
A behavioral approach to control of distributed systems,
SIAM J. Contr. Opt. 37:388–408, 1999.

References

M. Fliess and H. Mounier,
Controllability and observability of linear delay systems: an algebraic approach, ESAIM COCV, 3:301–314, 1998.

H. Mounier,
Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques,
PhD Thesis, University of Orsay, France, 1995.

J. Wood,
Modules and behaviours in nD systems theory,
Multidimensional Dimensional Systems and Signal Processing, 11:11–48, 2000.

A. Quadrat,
An introduction to constructive algebraic analysis and its applications,
Les cours du CIRM, 1 no. 2 (2010), pp. 281–471, INRIA Research Report no. 7354,
`http://hal.archives-ouvertes.fr/inria-00506104/fr/`.

References

V. Lomadze and E. Zerz,
Control and interconnection revisited: the linear multidimensional case, in: 2nd Int. Workshop on Multidimensional (ND) Systems, Czocha Castle, 2000.

E. Zerz,
Topics in Multidimensional Linear Systems Theory,
LNCIS 256, Springer, 2000.

F. Dubois and N. Petit and P. Rouchon,
Motion Planning and Nonlinear Simulations for a Tank Containing a Fluid, in: Proc. ECC, Karlsruhe (Germany), 1999.

N. Petit and P. Rouchon,
Dynamics and solutions to some control problems for water-tank systems,
IEEE Trans. Autom. Contr. 47 (4): 594–609, 2002.