STEADY STATE LARGE DEVIATIONS FOR ONE-DIMENSIONAL, SYMMETRIC EXCLUSION PROCESSES IN WEAK CONTACT WITH RESERVOIRS

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Abstract. Consider the symmetric exclusion process evolving on an interval and weakly interacting at the end-points with reservoirs. Denote by $I_{[0,T]}(\cdot)$ its dynamical large deviations functional and by $V(\cdot)$ the associated quasi-potential, defined as $V(\gamma) = \inf_{T>0} \inf_u I_{[0,T]}(u)$, where the infimum is carried over all trajectories u such that $u(0) = \bar{\rho}$, $u(T) = \gamma$, and $\bar{\rho}$ is the stationary density profile. We derive the partial differential equation which describes the evolution of the optimal trajectory, and deduce from this result the formula obtained by Derrida, Hirschberg and Sadhu [14] for the quasi-potential through the representation of the steady state as a product of matrices.

1. Introduction

Non-equilibrium steady states have attracted a lot of interest in the last decades, as a first step towards the understanding of far from equilibrium behavior. We refer to the reviews [12, 13, 6], the recent works [14, 20] and references therein. These states display many interesting phenomena, such as non-local thermodynamic functionals, dynamical phase transitions and long range correlations, [15, 9, 5]. Many of these properties can be derived from the quasi-potential, the functional which plays a role analogous to the free energy in equilibrium.

We consider the symmetric exclusion process evolving in the interval $[0, 1]$ and in contact with reservoirs at the end points. In the case of a strong interaction of the system with the reservoirs, the boundary conditions do not appear in the thermodynamic functionals and the effect of the boundary is not clear. To investigate the influence of the boundaries, we examine in this article the case of weak interactions.

With strong interactions with the reservoirs, the density at the boundaries take immediately the value of the reservoirs densities and remain fixed, while the density in the bulk evolves according to the heat equation. That is to say, the density profile evolves according to the heat equation with Dirichlet boundary conditions. Even at the level of the dynamical large deviations, the densities at the boundary are kept fixed, and only the density at the interior may fluctuate, [21, 3, 4].

In constrast, for exclusion processes with weak interactions with the reservoirs, the densities at the boundaries evolve in time. Actually, the particles' density u

solves the heat equation with Robin boundary conditions [2]. Namely,

$$\begin{cases} \partial_t u \,=\, \Delta u \quad (t,x) \,\in\, (0,T) \,\times\, (0,1) \\ (\nabla u)(t,0) \,=\, A^{-1}[\, u(t,0) - \alpha \,] \quad t \,\in\, (0,T) \\ (\nabla u)(t,1) \,=\, B^{-1}[\, \beta - u(t,1)] \quad t \,\in\, (0,T) \\ u(0,x) \,=\, \gamma(x) \quad x \,\in\, [0,1] \,. \end{cases} \tag{1.1}$$

In this formula, α, $\beta \in (0,1)$ represent the density at the left, right reservoirs, respectively, A, $B > 0$ the intensity of the interaction with the left, right reservoirs, respectively, and $\gamma : [0,1] \to [0,1]$ the initial density profile. Moreover, ∇u stands for the partial derivative in space of u, $\partial_t u$ for its partial derivative in time and Δu for the Laplacian of u in the space variable.

The weak interaction of the system with the boundaries also modifies the thermodynamical variables by adding boundary terms. The Hamiltonian, denoted by $\mathcal{H}(\gamma, F)$, becomes

$$\begin{aligned} \mathcal{H}(\gamma, F) \;=\; & -\, \big\langle \nabla\gamma \,,\, \nabla F \big\rangle \,+\, \big\langle \sigma(\gamma) \,,\, \big(\nabla F\big)^2 \big\rangle \\ & +\, \mathfrak{b}_{\alpha,A}\big(\, \gamma(0) \,,\, F(0) \,\big) \,+\, \mathfrak{b}_{\beta,B}\big(\, \gamma(1) \,,\, F(1) \,\big) \,. \end{aligned} \tag{1.2}$$

In this formula and below, $\langle \cdot \,,\, \cdot \rangle$ represents the scalar product in $\mathcal{L}^2([0,1])$, $\sigma : [0,1] \to \mathbb{R}$, given by $\sigma(a) = a(1-a)$, is the mobility of the exclusion process, and for $0 < \varrho < 1$, $D > 0$, $0 < a < 1$, $M \in \mathbb{R}$,

$$\mathfrak{b}_{\varrho,D}(a, M) \;=\; \frac{1}{D} \Big\{ [1-a]\, \varrho\, [e^{M} - 1] \;+\; a\, [1-\varrho]\, [e^{-M} - 1] \Big\} \,. \tag{1.3}$$

In the Hamiltonian formalism of classical mechanics, density profiles $\gamma : [0,1] \to [0,1]$ play the role of position and external fields $F : [0,1] \to \mathbb{R}$, the one of momentum.

The dynamical large deviations functional associated to the Hamiltonian $\mathcal{H}$ is given by

$$I_{[0,T]}(u) \;=\; \sup_{H} \int_0^T \big\{ \, \big\langle \partial_t u_t, H_t \big\rangle \,-\, \mathcal{H}(u_t, H_t) \, \big\} \, dt \,, \tag{1.4}$$

where the supremum is carried over all smooth functions $H : [0,T] \times [0,1] \to \mathbb{R}$. The functional $I_{[0,T]}(u)$ specifies the cost of observing a fluctuation $u(t)$, $0 \le t \le T$. In particular, $I_{[0,T]}(u) = 0$ if u follows the hydrodynamic equation (1.1).

Let $\bar\rho$ be the unique stationary solution to the equation (1.1). That is, $\bar\rho$ is the solution to the elliptic equation

$$\begin{cases} \Delta\rho \,=\, 0 \\ (\nabla\rho)(0) \,=\, A^{-1}[\, \rho(0) - \alpha \,] \\ (\nabla\rho)(1) \,=\, B^{-1}[\, \beta - \rho(1)] \,. \end{cases} \tag{1.5}$$

An elementary computation yields that $\bar\rho$ is given by

$$\bar\rho(x) \;=\; \frac{\alpha(1+B) + \beta A}{1+B+A} \;+\; \frac{(\beta-\alpha)\, x}{1+B+A} \,.$$

Note that $\bar\rho$ is the linear interpolation between $\bar\rho(-A) = \alpha$ and $\bar\rho(1+B) = \beta$.

Denote by $V(\,\cdot\,)$ the quasi-potential associated to the rate function $I_{[0,T]}(\,\cdot\,)$. It is given by

$$V(\gamma) \;:=\; \inf_{T>0} \, \inf_{u(\cdot)} \; I_{[0,T]}(u) \,,$$

where the infimum is carried over all paths u such that $u(0) = \bar\rho$, $u(T) = \gamma$. The quasi-potential $V(\gamma)$ measures the minimal cost to produce a profile γ starting from $\bar\rho$. It is also the rate functional of the large deviations principle for the density profile under the steady state [3].

By using a representation of the steady state of the exclusion process as a product of matrices, Derrida, Hirschberg and Sadhu [14] proved that the quasi-potential can be expressed as

$$\begin{aligned} V(\gamma) \;=\; \int_0^1 \Big\{ \gamma(x) \, \log \frac{\gamma(x)}{F(x)} + \big[1-\gamma(x)\big] \log \frac{1-\gamma(x)}{1-F(x)} + \log \frac{\nabla F(x)}{[\beta-\alpha]} \Big\} \, dx \\ + A \, \ln \frac{F(0)-\alpha}{A(\beta-\alpha)} \;+\; B \, \ln \frac{\beta - F(1)}{B(\beta-\alpha)} \;, \end{aligned}$$

where F solves the non linear boundary value problem

$$\begin{cases} \Delta F = \big(\gamma - F\big) \dfrac{\big(\nabla F\big)^2}{F(1-F)} & \text{in } (0,1) \;, \\ \nabla F(0) = A^{-1}[F(0)-\alpha] \;, \quad \nabla F(1) = B^{-1}[\beta - F(1)] \;. \end{cases} \tag{1.6}$$

This result extends to exclusion processes with weak interactions at the boundaries a theorem of Derrida, Lebowitz and Speer [15] for the case with strong interactions,

In this article, we provide an alternative proof of this result, based on the strategy delineated in [3] and carried out in [4] for exclusion processes with strong interactions at the boundaries.

Using the Hamiltonian formalism, we derive a formal equation for the path which solves the variational problem (1.4). The optimal trajectory corresponds to a pair $(u(t), F(t))$ which solves a system of coupled equations, see (3.5)–(3.6) below. While it might seem, at a first glance, hopeless to prove any property of the solutions to this pair of equations, it turns out that F_t evolves according to an autonomous equation, actually, according to the hydrodynamic equation (1.1). This remarkable property is the key point and permits to prove all properties needed to show that the candidate obtained from the heuristic argument is indeed the optimal path.

According to [3], this optimal path, which describe how the system adjusts to create a fluctuation of the density, corresponds to the typical trajectory for the adjoint dynamics, reversed in time. In particular, this approach reveals the adjoint hydrodynamic equation. It is obtained from the hydrodynamic equation by adding a non-local drift and modifying the boundary densities and intensities of interaction which become time-dependent (cf. Remark 3.1).

To our knowledge this is one of the few examples of an interacting particle system whose steady state is not known explicitly and whose quasi-potential can be computed [15, 4, 1, 14]. Moreover, the presence of boundary terms in the Hamiltonian-Jacobi equation for the quasi-potential modifies entirely the analysis of thermodynamic transformations of non-equilibrium states carried out in [7] in the case of Dirichlet boundary conditions. This is left for a future work, together with the emergence of dynamical phase transitions [5] and the static large deviations [10, 17, 22, 18].

2. Notation and Results

The model. We consider one-dimensional, symmetric exclusion processes in weak contact with boundary reservoirs. Fix $N \ge 1$, and let $\mathfrak{e}_N = 1/N$, $\mathfrak{r}_N = 1 - (1/N)$, $\Lambda_N = \{\mathfrak{e}_N, \dots, (N-2)\,\mathfrak{e}_N, \mathfrak{r}_N\}$. The sate space is represented by $\Omega_N = \{0,1\}^{\Lambda_N}$ and the configurations by the Greek letters η, ξ so that η_x, $x \in \Lambda_N$, represents the number of particles at site x for the configuration η.

Fix throughout this article, $0 < \alpha \le \beta < 1$, $A > 0$, $B > 0$. The generator of the Markov process considered here, represented by $\mathcal{L}_N = \mathcal{L}_N^{\alpha, A, \beta, B}$, is given by

$$\mathcal{L}_N \;=\; L_N^{\rm lb} \;+\; L_N^{\rm bulk} \;+\; L_N^{\rm rb}\;.$$

In this formula, for every function $f : \Omega_N \to \mathbb{R}$,

$$(L_N^{\rm bulk} f)(\eta) \;=\; N^2 \sum_{x\in \Lambda_N^o} \big[\, f(\sigma^{x,x+\mathfrak{e}_N}\eta) \,-\, f(\eta)\,\big]\;,$$

where Λ_N^o represents the interior of Λ_N, $\Lambda_N^o := \Lambda_N \setminus \{\mathfrak{r}_N\} = \{\mathfrak{e}_N, \dots, (N-2)\,\mathfrak{e}_N\}$, and

$$\begin{aligned} (L_N^{\rm lb} f)(\eta) \;&=\; \frac{N}{A}\, \big[\,(1-\eta_{\mathfrak{e}_N})\,\alpha \,+\, (1-\alpha)\,\eta_{\mathfrak{e}_N}\,\big] \big[\, f(\sigma^{\mathfrak{e}_N}\eta) \,-\, f(\eta)\,\big]\;, \\ (L_N^{\rm rb} f)(\eta) \;&=\; \frac{N}{B}\, \big[\,(1-\eta_{\mathfrak{r}_N})\,\beta \,+\, (1-\beta)\,\eta_{\mathfrak{r}_N}\,\big] \big[\, f(\sigma^{\mathfrak{r}_N}\eta) \,-\, f(\eta)\,\big]\;. \end{aligned}$$

From now on, we omit the subindex N of $\mathfrak{e}_N$. In the formulas above,

$$(\sigma^{x,x+\mathfrak{e}}\eta)_y \;=\; \begin{cases} \eta_y & \text{if } y \neq x, x+\mathfrak{e} \\ \eta_{x+\mathfrak{e}} & \text{if } y = x \\ \eta_x & \text{if } y = x+\mathfrak{e} \end{cases} \qquad \text{and} \qquad (\sigma^x\eta)_y \;=\; \begin{cases} \eta_y & \text{if } y \neq x \\ 1-\eta_x & \text{if } y = x\;. \end{cases}$$

For a metric space $\mathbb{X}$, denote by $D([0,T], \mathbb{X})$, $T > 0$, the space of right-continuous functions $\mathfrak{x}\colon [0,T] \to \mathbb{X}$, with left-limits, endowed with the Skorohod topology and its associated Borel σ-algebra. The elements of $D([0,T], \Omega_N)$ are represent by $\boldsymbol{\eta}(\cdot)$.

For a probability measure μ on Ω_N, let $\mathbb{P}^N_\mu$ be the measure on $D([0,T], \Omega_N)$ induced by the continuous-time Markov process associated to the generator $\mathcal{L}_N$ starting from μ. When the measure μ is the Dirac measure concentrated at a configuration $\eta \in \Omega_N$, that is $\mu = \delta_\eta$, we represent $\mathbb{P}^N_{\delta_\eta}$ simply by $\mathbb{P}^N_\eta$. Expectation with respect to $\mathbb{P}^N_\mu$, $\mathbb{P}^N_\eta$ is denoted by $\mathbb{E}^N_\mu$, $\mathbb{E}^N_\eta$, respectively. When the context permits we remove the index N from the notation.

Hydrodynamic limit. Denote by $\mathcal{M}$ the set of non-negative measures on $[0,1]$ with total mass bounded by 1 endowed with the weak topology. Recall that this topology is metrisable and that, with this topology, $\mathcal{M}$ is a relatively compact space. For a continuous function $F : [0,1] \to \mathbb{R}$ and a measure $\pi \in \mathcal{M}$, denote by $\langle \pi, F\rangle$ the integral of F with respect to μ:

$$\langle \pi, F\rangle \;=\; \int F(x)\,\pi(dx)\;.$$

Given a configuration $\eta \in \Omega_N$, denote by $\pi = \pi(\eta)$ the measure in $\mathcal{M}$ obtained by assigning a mass N^{-1} to the position of each particle:

$$\pi \;=\; \pi(\eta) \;=\; \frac{1}{N} \sum_{x\in\Lambda_N} \eta_x\,\delta_x\;.$$

The measure π is called the *empirical measure*.

Denote by $\boldsymbol{\pi} : D([0,T],\Omega_N) \to D([0,T],\mathcal{M})$ the map which associates to a trajectory $\boldsymbol{\eta}(\cdot)$ its empirical measure:

$$\boldsymbol{\pi}(t) \;=\; \pi(\boldsymbol{\eta}(t)) \;=\; \sum_{x\in\Lambda_N} \eta_x(t)\,\delta_x \;.$$

For a probability measure μ in Ω_N, let $\mathbb{Q}^N_\mu$ be the measure on $D([0,T],\mathcal{M})$ given by $\mathbb{Q}^N_\mu = \mathbb{P}^N_\mu \circ \boldsymbol{\pi}^{-1}$.

The first result, due to [2], establishes the hydrodynamic behavior of the empirical measure.

Theorem 2.1. *Fix $T>0$, a density profile $\gamma\colon[0,1]\to[0,1]$, and sequence $(\nu^N : N\ge 1)$ of probability measures on Ω_N associated to γ in the sense that*

$$\lim_{N\to\infty} \nu^N\Big[\,\Big|\,\langle \pi\,,\,G\rangle \;-\; \int_0^1 \gamma(x)\,G(x)\,dx\,\Big| \;>\;\delta\,\Big] \;=\; 0$$

for all continuous functions $G\colon[0,1]\to\mathbb{R}$ and $\delta>0$. Then, the sequence of probability measures $\mathbb{Q}^N_{\nu^N}$ converges to the probability measure $\mathbb{Q}$ concentrated on the trajectory $\pi(t,dx)=u(t,x)\,dx$, where u is the unique weak solution to the heat equation with Robin's boundary conditions (1.1).

We refer to Appendix B for the definition of weak solutions to equation (1.1) and some of its properties.

Dynamical large deviations. For $T>0$ and positive integers m,n, denote by $C^{m,n}([0,T]\times[0,1])$ the space of functions $G\colon[0,T]\times[0,1]\to\mathbb{R}$ with m derivatives in time, n derivatives in space which are continuous up to the boundary. Denote by $C^{m,n}_0([0,T]\times[0,1])$ the set of functions in $C^{m,n}([0,T]\times[0,1])$ which vanish at the endpoints of $[0,1]$, i.e. $G\in C^{m,n}([0,T]\times[0,1])$ belongs to $C^{m,n}_0([0,T]\times[0,1])$ if and only if $G(t,0)=G(t,1)=0$ for all $t\in[0,T]$.

Denote by $\mathcal{M}_{\rm ac}$ the subset of $\mathcal{M}$ of all measures which are absolutely continuous with respect to the Lebesgue measure and whose density takes values in the interval $[0,1]$: $\mathcal{M}_{\rm ac} = \{\pi\in\mathcal{M} : \pi(dx)=\gamma(x)\,dx \;\text{ and }\; 0\,\le\,\gamma(x)\,\le\,1\,\}$.

For $T>0$, let the *energy* $\mathcal{Q}_{[0,T]} : D([0,T],\mathcal{M}_{\rm ac})\to[0,\infty]$ be given by

$$\mathcal{Q}_{[0,T]}(\pi) \;=\; \sup_G \Big\{ \int_0^T dt \int_0^1 u(t,x)\,(\nabla G)(t,x)\;dx \;-\; \frac{1}{2}\int_0^T dt\int_0^1 \sigma(u(t,x))\,G(t,x)^2\;dx\,\Big\}\;,$$

where $\pi(t,dx)=u(t,x)\,dx$ and the supremum is carried over all smooth functions $G:[0,T]\times(0,1)\to\mathbb{R}$ with compact support.

Remark 2.2. *Hereafter, we abuse of notation writing $\gamma\in\mathcal{M}_{\rm ac}$ to mean that the measure $\gamma(x)\,dx$ belongs to $\mathcal{M}_{\rm ac}$. Moreover, for functionals $\Phi\colon D([0,T],\mathcal{M}_{\rm ac})\to\mathbb{R}$, $W:\mathcal{M}_{\rm ac}\to\mathbb{R}$, we often write $\Phi(u)$, $W(\gamma)$ instead of $\Phi(\pi)$, $W(\mu)$ when $\pi(t,dx)=u(t,x)\,dx$, $\mu(dx)=\gamma(x)\,dx$.*

Notational convention: For a function $v:I\times[0,1]\to\mathbb{R}$, where I is a subset of $\mathbb{R}$, v_t and $v(t)$, $t\in I$, represent the function $w:[0,1]\to\mathbb{R}$ defined by $w(x)=v(t,x)$. We use the letters γ, ϕ, ψ to represent densities [elements of $\mathcal{M}_{\rm ac}$], u, v, w for trajectories of densities [elements of $D(\mathbb{R}_+,\mathcal{M}_{\rm ac})$], and F, G, H for external fields, usually functions in $C(\mathbb{R}_+\times[0,1])$.

By [8, Lemma 4.1], the energy $\mathcal{Q}_{[0,T]}$ is convex and lower semicontinuous. Moreover, if $\mathcal{Q}_{[0,T]}(u)$ is finite, u has a generalized space derivative, denoted by ∇u, and

$$\mathcal{Q}_{[0,T]}(u) \;=\; \frac{1}{2}\int_0^T dt \int_0^1 \frac{(\nabla u_t)^2}{\sigma(u_t)}\, dx \cdot$$

Fix a trajectory $\pi \in D([0,T],\mathcal{M}_{\rm ac})$, $\pi(t,dx) = u(t,x)\,dx$, with finite energy, $\mathcal{Q}_{[0,T]}(u) < \infty$. In particular, $\int_0^T dt \int_0^1 (\nabla u_t)^2 \, dx$ is finite. By [27, Assertion 48, page 1030], the trace of u at the spatial boundary of the cylinder $\Omega_T = [0,T]\times[0,1]$ is well defined. That is, the maps $t\mapsto u(t,0)$, $t\mapsto u(t,1)$ are well defined and belong to $\mathcal{L}^2([0,T])$. Moreover, since for almost all $t\in[0,T]$, $\int_0^1 (\nabla u_t)^2\, dx$ is finite, for these values of t, $u(t,\cdot)$ is Hölder-continuous and $u(t,0)$ and $u(t,1)$ are well defined.

Denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the usual scalar product in $\mathcal{L}^2([0,1])$:

$$\langle\, f\,,\, g\,\rangle \;=\; \int_0^1 f(x)\, g(x)\, dx\;, \quad f\,,\, g \,\in\, \mathcal{L}^2([0,1])\;.$$

Fix a function $\gamma\colon [0,1]\to[0,1]$, which corresponds to the initial profile. Denote by $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ the set of trajectories in $D([0,T],\mathcal{M}_{\rm ac})$ with finite energy, and by $D_{\gamma,\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ the set of trajectories with finite energy and which start from γ, $u_0(\cdot)=\gamma(\cdot)$ a.s.

Recall, from (1.3), the definition of $\mathfrak{b}_{\varrho,D}(a,M)$ and, from Remark 2.2, the convention on notation. For each H in $C^{1,2}([0,T]\times[0,1])$, let $J_{T,H}\colon D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac}) \longrightarrow \mathbb{R}$ be the functional given by

$$\begin{aligned} J_{T,H}(u) \;=\; & \big\langle u_T, H_T\big\rangle \;-\; \langle u_0, H_0\rangle \;-\; \int_0^T \big\langle u_t, \partial_t H_t\big\rangle\, dt \\ -\; & \int_0^T \big\langle u_t, \Delta H_t\big\rangle\, dt \;+\; \int_0^T u_t(1)\,\nabla H_t(1)\, dt \;-\; \int_0^T u_t(0)\,\nabla H_t(0)\, dt \\ -\; & \int_0^T \big\langle \sigma(u_t), \big(\nabla H_t\big)^2\big\rangle\, dt \\ -\; & \int_0^T \Big\{\, \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\, u_t(1)\,,\, H_t(1)\,\big)\,\Big\}\, dt\;. \end{aligned} \tag{2.1}$$

The right-hand side is well defined because the functions $u(\cdot,0)$, $u(\cdot,1)$ belong to $\mathcal{L}^2([0,T])$.

Since trajectories in $D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})$ have generalized space-derivatives, we may integrate by parts the second line and write the functional $J_{T,H}(\,\cdot\,)$ as

$$\begin{aligned} J_{T,H}(u) \;=\; & \big\langle u_T, H_T\big\rangle \;-\; \langle u_0, H_0\rangle \;-\; \int_0^T \big\langle u_t, \partial_t H_t\big\rangle\, dt \\ & +\; \int_0^T \big\langle \nabla u_t, \nabla H_t\big\rangle\, dt - \int_0^T \big\langle \sigma(u_t), \big(\nabla H_t\big)^2\big\rangle\, dt \\ & -\; \int_0^T \Big\{\, \mathfrak{b}_{\alpha,A}\big(\, u_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\, u_t(1)\,,\, H_t(1)\,\big)\,\Big\}\, dt\;. \end{aligned} \tag{2.2}$$

Let $I_{[0,T]}\colon D_{\mathcal{E}}([0,T],\mathcal{M}_{\rm ac}) \to [0,+\infty]$ be the functional defined by

$$I_{[0,T]}(\pi) \;:=\; \sup_{H\in C^{1,2}([0,T]\times[0,1])} J_{T,H}(\pi)\;.$$

Fix a density profile γ in $\mathcal{M}_{\rm ac}$, and let $I_{[0,T]}(\,\cdot\,|\gamma)\colon D([0,T],\mathcal{M})\to\mathbb{R}$ be given by

$$I_{[0,T]}(\pi|\gamma) \;=\; \begin{cases} I_{[0,T]}(\pi) & \text{if } \pi \,\in\, D_{\gamma,\mathcal{E}}([0,T],\mathcal{M}_{\rm ac})\;, \\ \infty & \text{otherwise}\;. \end{cases} \tag{2.3}$$

We review in Section 4 some properties of the functional $I_{[0,T]}(\cdot|\gamma)$ obtained in [19]. Next result is the main theorem in [19].

Theorem 2.3. *Fix $T>0$ and a measure $\pi(dx)=\gamma(x)\,dx$ in $\mathcal{M}_{\rm ac}$. Consider a sequence η^N of configurations associated to γ. Then the measure $\mathbb{Q}_{\eta^N}$ satisfies a large deviation principle with speed N and rate function $I_{[0,T]}(\cdot|\gamma)$. Namely, for each closed set $\mathcal{C}\subset D([0,T],\mathcal{M})$ and each open set $\mathcal{O}\subset D([0,T],\mathcal{M})$,*

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}[\boldsymbol{\pi}\in\mathcal{C}] \;\le\; -\inf_{\pi\in\mathcal{C}} I_{[0,T]}(\pi|\rho)$$
$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}_{\eta^N}[\boldsymbol{\pi}\in\mathcal{O}] \;\ge\; -\inf_{\pi\in\mathcal{O}} I_{[0,T]}(\pi|\rho)\;.$$

The quasi-potential. Denote by $V:\mathcal{M}_{\rm ac}\to\mathbb{R}_+$ the quasi-potential associated to the rate function $I_{[0,T]}(\,\cdot\,|\,\gamma)$. It is given by

$$V(\gamma) \;:=\; \inf_{T>0}\,\inf_{u(\cdot)}\; I_{[0,T]}(u\,|\,\bar\rho)\;, \tag{2.4}$$

where the infimum is carried over all paths u in $D([0,T],\mathcal{M}_{\rm ac})$ such that $u(0)=\bar\rho$, $u(T)=\gamma$. The quasi-potential $V(\gamma)$ measures the minimal cost to produce a profile γ starting from $\bar\rho$.

Denote by $C^1\big([0,1]\big)$ the space of once continuously differentiable functions $F:[0,1]\to\mathbb{R}$ endowed with the norm $\|F\|_{C^1}:=\sup_{x\in[0,1]}\big\{|F(x)|+|\nabla F(x)|\big\}$. Let $\mathcal{F}$ be the space of monotone C^1 functions:

$$\mathcal{F}:=\big\{\,F\in C^1\big([0,1]\big)\,:\,\alpha\,<\,F(x)\,<\,\beta\,,\;\nabla F(x)>0\;\;\forall\,x\in[0,1]\,\big\}\;. \tag{2.5}$$

Denote by $\mathcal{G}_{\rm bulk}$, $\mathcal{G}\colon\mathcal{M}_{\rm ac}\times\mathcal{F}\to\mathbb{R}$ the functionals given by

$$\mathcal{G}_{\rm bulk}(\gamma,F):=\int_0^1\Big\{\,\gamma(x)\,\log\frac{\gamma(x)}{F(x)}+\big[1-\gamma(x)\big]\log\frac{1-\gamma(x)}{1-F(x)}+\log\frac{\nabla F(x)}{[\beta-\alpha]}\,\Big\}\,dx\;,$$
$$\mathcal{G}(\gamma,F)\;:=\;\mathcal{G}_{\rm bulk}(\gamma,F)\;+\;A\,\ln\frac{F(0)-\alpha}{A(\beta-\alpha)}\;+\;B\,\ln\frac{\beta-F(1)}{B(\beta-\alpha)}\;.$$

Define S_0, $S\colon\mathcal{M}_{\rm ac}\to\mathbb{R}$, by

$$S_0(\gamma)\;:=\;\sup_{F\in\mathcal{F}}\mathcal{G}(\gamma,F)\;,\quad S(\gamma)\;:=\;S_0(\gamma)\,-\,S_0(\bar\rho)\;. \tag{2.6}$$

Main results. The first main assertion of the article, Theorem 5.2, affirms that, for each $\gamma\in\mathcal{M}_{\rm ac}$, the non-linear boundary-value problem (1.6) has a unique solution in $\mathcal{F}$. Its precise statement is left to Section 5 because it requires some notation.

Theorem 2.4. *The functional $S\colon\mathcal{M}_{\rm ac}\to\mathbb{R}$ defined in* (2.6) *is bounded, convex and lower semi-continuous. Moreover, $S_0(\gamma)=\mathcal{G}(\,\gamma,F(\gamma)\,)$, where $F(\gamma)$ is the solution to* (1.6).

Remark 2.5. *When $\gamma=\bar\rho$, $F=\bar\rho$ is the solution to* (1.6). *Replacing F by $\bar\rho$ in the formula for $\mathcal{G}$ yields that $S_0(\bar\rho)\;=\;-\,(1+A+B)\,\log(1+A+B)$.*

Next theorem asserts that the functionals defined through the variational problems (2.4) and (2.6) coincide. In particular, it gives an "explicit" formula for the dynamical variational problem (2.4) defining the quasi-potential.

Theorem 2.6. *For each* $\gamma \in \mathcal{M}_{\rm ac}$, $V(\gamma) = S(\gamma)$. *In particular, the functional* S *is non-negative. That is, the functional* S_0 *attains its minimum at* $\bar\rho$.

Remark 2.7. *Theorems 2.4 and 2.6 formalize the arguments presented in* [14], *where Derrida, Hirschberg and Sadhu derived the steady state large deviations functional by representing the steady state as a product of matrices.*

Remark 2.8. *If* $\alpha = \beta$, *the exclusion dynamics is reversible and the stationary state is the Bernoulli product measure with density* α. *In particular, in this case*

$$S(\gamma) \;=\; \int_0^1 \Big\{ \gamma(x)\, \log \frac{\gamma(x)}{\alpha} + \big[1-\gamma(x)\big] \log \frac{1-\gamma(x)}{1-\alpha} \Big\}\, dx\;.$$

The same strategy as in the proof of Theorem 2.6 yields that $V = S$. *The arguments are much simpler because the adjoint dynamics coincides with the original one as the process is reversible.*

The method of the proof of Theorems 2.4 and 2.6 is the one proposed in [3] and carried out in [4] for exclusion processes with strong interaction with the reservoirs.

The fact that the densities are not fixed by the dynamics at the boundary and the presence of exponential terms at the boundary in the dynamical large deviations functionals (cf. the definition of $\mathfrak{b}_{\varrho,D}$), introduce many new difficulties. The proof of the uniqueness of solutions to (1.6) is one of them (cf. Proof of Theorem 5.2).

The proof of the upper bound for the quasi-potential is a second example. As the boundary conditions are not fixed, to prove that the solutions to the adjoint hydrodynamic equations are bounded away from 0 and 1, we need to investigate the behavior of the solutions at the boundary. This is done in the proof of Proposition 6.7.

The article is organized as follows. In Section 3 we present a heuristic derivation of Theorem 2.6 based on the Hamiltonian formalism of rational mechanics. This argument explains the strategy adopted in the following sections. In Section 4, we recall some properties of the dynamical rate function $I_{[0,T]}(\,\cdot\,|\gamma)$ obtained in [19]. Theorems 2.4, 2.6 are proved in Sections 5, 6, respectively. In Appendices A and B we present some results on the Robin Laplacian and on solutions to heat equations with mixed boundary conditions needed in the proofs of the main theorems.

3. Sketch of the proof of Theorem 2.6.

The arguments below are formal, but explain the idea of the proof. We follow the strategy proposed in [4], in the context of boundary driven symmetric simple exclusion processes with strong interaction with the boundaries, to derive a formula for the quasi-potential based on the Hamiltonian formalism. We also introduce the hydrodynamic equation of the adjoint process, which describes how the dynamics acts to create an anomalous density profile. This section also serves as a road map to prove Theorem 2.6 in other contexts.

Recall from (1.2) the definition of the Hamiltonian $\mathcal{H}$. With this notation and an integration by parts in time, the functional $I_{[0,T]}$ can be written as

$$I_{[0,T]}(\pi|\gamma) \;=\; \sup_H \int_0^T \big\{ \big\langle \partial_t u_t , H_t \big\rangle \,-\, \mathcal{H}(u_t, H_t) \big\}\, dt\;. \tag{3.1}$$

Hence, the functional $I_{[0,T]}(\,\cdot\,|\gamma)$ corresponds to the action functional associated to the Hamiltonian $\mathcal{H}$.

A variational calculation yields that the quasi-potential satisfies the Hamilton-Jacobi equation: for every $\gamma \in \mathcal{M}_{\rm ac}$,

$$\mathcal{H}\big(\,\gamma\,,\,\frac{\delta V}{\delta \gamma}\,\big) \;=\; 0\,. \tag{3.2}$$

where $\delta V/\delta\gamma$ stands for the functional derivative of V.

Fix $\gamma \in \mathcal{M}_{\rm ac}$ and let $\Gamma = \log[\gamma/(1-\gamma)] - \log[F/(1-F)]$ for some function F taking values in the interval $(0,1)$. Lemma 6.2 asserts that, if γ is smooth and bounded away from 0 and 1, Γ solves the Hamilton-Jacobi equation

$$\mathcal{H}(\gamma, \Gamma) \;=\; 0 \tag{3.3}$$

if F is the solution to (1.6), that is, if $F = F(\gamma)$ with the notation introduced in the statement of Theorem 2.4. By (3.2) and (3.3),

$$\frac{\delta V}{\delta\gamma} \;=\; \log\frac{\gamma}{1-\gamma} \;-\; \log\frac{F}{1-F}\;\cdot \tag{3.4}$$

To build a functional S which satisfies (3.4), we look for a functional $\mathcal{W}(\gamma, F)$, with two properties:

(a) For every $\gamma \in \mathcal{M}_{\rm ac}$,

$$\frac{\delta \mathcal{W}}{\delta\gamma}(\gamma, F) \;=\; \log\frac{\gamma}{1-\gamma} \;-\; \log\frac{F}{1-F}\;,$$

(b) For each $\gamma \in \mathcal{M}_{\rm ac}$, the solution $F(\gamma)$ of equation (1.6) is a critical point of $\mathcal{W}(\gamma,\cdot)$;

Under these assumptions, defining $S(\gamma)$ as $\mathcal{W}(\gamma, F(\gamma))$, we have

$$\frac{\delta S}{\delta\gamma}(\gamma) \;=\; \frac{\delta \mathcal{W}}{\delta\gamma}(\gamma, F(\gamma)) \;+\; \frac{\delta \mathcal{W}}{\delta F}(\gamma, F(\gamma))\,\frac{\delta F}{\delta\gamma}(\gamma) \;=\; \frac{\delta \mathcal{W}}{\delta\gamma}(\gamma, F(\gamma))\;\cdot$$

The last identity follows from property (b) of the functional $\mathcal{W}$ $[\delta\mathcal{W}/\delta F = 0$ at $(\gamma, F(\gamma))]$. By property (a), the right-hand side is equal to $\log[\gamma/(1-\gamma)] - \log[F(\gamma)/(1-F(\gamma))]$, proving that (3.4) is fulfilled.

This computation explains the introduction of the functional $\mathcal{G}(\gamma, F)$, defined below (2.6). It is obtained by integrating (3.4) in γ and adding terms which depend only on F to match condition (b). The functional $\mathcal{G}$ satisfies properties (a) and (b), as it is easy to show that (1.6) corresponds to the Euler-Lagrange equation of the functional $\mathcal{G}(\gamma,\,\cdot\,)$.

We turn to the proof that $V = S$. Fix $\gamma \in \mathcal{M}_{\rm ac}$, $T > 0$ and a trajectory u_t, $0 \le t \le T$, such that $u_0 = \bar\rho$, $u_T = \gamma$. Let F_t be the solution to (1.6) with u_t replacing γ, $F_t = F(u_t)$. By (3.1), (3.3),

$$I_{[0,T]}(\pi|\gamma) \;\ge\; \int_0^T \big\langle \partial_t u_t, \Gamma_t \big\rangle\, dt\;,$$

where $\Gamma_t = \log[u_t/(1-u_t)] - \log[F_t/(1-F_t)]$. In view of (3.4), replacing Γ_t by $(\delta S/\delta\gamma)(u_t)$ yields that

$$I_{[0,T]}(\pi|\gamma) \;\ge\; S(u_T) \,-\, S(u_0) \;=\; S(\gamma) \,-\, S(\bar\rho)\;,$$

so that

$$V(\gamma) \;\ge\; S(\gamma) \,-\, S(\bar\rho)\;.$$

We proceed with the upper bound. By [3], the optimal trajectory for the variational problem (2.4) is the hydrodynamic trajectory of the adjoint dynamics reversed in time. Moreover, according to [3], the adjoint dynamics is given by

$$\partial_t v \;=\; -\,\Delta v \;+\; 2\,\nabla\big(\,\sigma(v)\,\nabla\frac{\delta S}{\delta v}\,\big)$$

In view of (3.4), replacing $\delta S/\delta v$ by $\log[v_t/(1-v_t)] - \log[F_t/(1-F_t)]$, where F_t is the solution to (1.6) with v_t in place of γ yields the equation

$$\partial_t v \;=\; \Delta v \;-\; 2\,\nabla\big(\,\sigma(v)\,\nabla\log\frac{F}{1-F}\,\big)\;.$$

Adding the boundary and initial conditions, as well as the equation for F, the previous equation becomes the system of equations

$$\begin{cases} \partial_t v_t \;=\; \Delta v_t \;-\; 2\,\nabla\big(\,\sigma(v_t)\,\nabla R_t\,\big) \quad (t,x)\in(0,\infty)\times(0,1)\;, \\ \nabla v_t(1) \;-\; 2\,\sigma(v_t(1))\,\nabla R_t(1) \;=\; \mathfrak{p}_{1-\beta,B}\big(\,v_t(1)\,,\,R_t(1)\,\big)\;, \\ \nabla v_t(0) \;-\; 2\,\sigma(v_t(0))\,\nabla R_t(0) \;=\; -\,\mathfrak{p}_{1-\alpha,A}\big(\,v_t(0)\,,\,R_t(0)\,\big)\;, \\ v_0(\cdot) \;=\; \gamma(\cdot)\;, \quad x\in[0,1]\;, \end{cases} \tag{3.5}$$

$$\begin{cases} \Delta F_t = \big(v_t - F_t\big)\dfrac{\big(\nabla F_t\big)^2}{F_t(1-F_t)} \quad (t,x)\in(0,\infty)\times(0,1)\;, \\ \nabla F_t(0) = A^{-1}[F_t(0)-\alpha]\;, \quad \nabla F_t(1) = B^{-1}[\beta - F_t(1)]\;. \end{cases} \tag{3.6}$$

In this formula, $R_t = \log[F_t/(1-F_t)]$ and, for $0<\varrho<1$, $D>0$, $0<a<1$, $M\in\mathbb{R}$,

$$\mathfrak{p}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\,\Big\{[1-a]\,\varrho\,e^{M} \;-\; a\,[1-\varrho]\,e^{-M}\Big\}\;. \tag{3.7}$$

The first part of the proof of the upper bound consists in showing that this trajectory is indeed the optimal one. Lemma 6.5, whose proof relies on the explicit expression for the rate functional presented in Lemma 4.4, states that this trajectory is optimal provided the solution $v(t)$ to this equation relaxes to $\bar\rho$ as $t\to\infty$.

To prove that v_t relaxes to $\bar\rho$ or any other property of the non-local system of equations (3.5)–(3.6) looks hopeless. It turns out, however, that these equations can be expressed in a simple form. The reason is that F_t in (3.5)–(3.6), which corresponds to the momentum in the Hamiltonian formalism, evolves according to an autonomous equation, a remarkable and unexpected property.

Fix $\gamma\in\mathcal{M}_{\rm ac}$, and denote by $F^{(\gamma)}$ the solution to (1.6). Let $F^{(\gamma)}_t$ be the solution to the heat equation (1.1) with initial condition $F^{(\gamma)}$ instead of γ. Define $v^{(\gamma)}_t$ as

$$v^{(\gamma)}(t) \;:=\; F^{(\gamma)}(t) \;+\; F^{(\gamma)}(t)\,[1-F^{(\gamma)}(t)]\,\frac{\Delta F^{(\gamma)}(t)}{\big(\nabla F^{(\gamma)}(t)\big)^2}\;\cdot \tag{3.8}$$

By (1.6), $v^{(\gamma)}(0)=\gamma$. Actually, $F^{(\gamma)}_t = F(v^{(\gamma)}_t)$ for all $t\ge 0$, where $F(v^{(\gamma)}_t)$ is the solution to (1.6) with γ replaced by $v^{(\gamma)}_t$.

Proposition 6.7 asserts that for each $\gamma\in C^1([0,1])$ the pair $(v^{(\gamma)}_t, F^{(\gamma)}_t)$ solves the system of equations (3.5)–(3.6). This result provides, therefore, an alternative and simple formulation of these equations. Moreover, by Lemma 6.10, $\lim_{t\to\infty} v^{(\gamma)}(t) = \bar\rho$, and, by Lemma 6.12, the optimal path which solves the variational problem (2.4), represented by $u^{(\gamma)}_{\rm opt}(t)$, defined on the time interval $(-\infty,0]$ instead of $[0,+\infty)$ as in

(2.4), is given by $u^{(\gamma)}_{\rm opt}(t) = v^{(\gamma)}(-t)$. Note that $u^{(\gamma)}_{\rm opt}(0) = \gamma$, $\lim_{t\to-\infty} u^{(\gamma)}_{\rm opt}(t) = \bar\rho$. Hence, $u^{(\gamma)}_{\rm opt}$ connects $\bar\rho$ to γ in a infinite time window.

Remark 3.1. *The boundary conditions in* (3.5) *can be written as*

$$\begin{cases} \nabla v_t(0) \,-\, 2\,\sigma(v_t(0))\,\nabla R_t(0) \,=\, \dfrac{1}{A^*_t}\,[\,v(t,0) - \alpha^*_t\,]\;. \\ \nabla v_t(1) \,-\, 2\,\sigma(v_t(1))\,\nabla R_t(1) \,=\, \dfrac{1}{B^*_t}\,[\,\beta^*_t - v(t,1)\,]\;, \end{cases} \tag{3.9}$$

where

$$\frac{1}{B^*_t} \;=\; \frac{1}{B}\left\{\,(1-\beta)\,e^{R_t(1)} \;+\; \beta\,e^{-R_t(1)}\,\right\}\,, \quad \beta^*_t \;=\; \frac{(1-\beta)\,e^{R_t(1)}}{(1-\beta)\,e^{R_t(1)} \;+\; \beta\,e^{-R_t(1)}}$$

$$\frac{1}{A^*_t} \;=\; \frac{1}{A}\left\{\,(1-\alpha)\,e^{R_t(0)} \;+\; \alpha\,e^{-R_t(0)}\,\right\}\,, \quad \alpha^*_t \;=\; \frac{(1-\alpha)\,e^{R_t(0)}}{(1-\alpha)\,e^{R_t(0)} \;+\; \alpha\,e^{-R_t(0)}}\;.$$

In view of (3.9)*, equation* (3.5) *corresponds to the hydrodynamic equation of the weakly asymmetric exclusion process with weak interactions at the boundary (cf.* [19]*). Note that this equation carries a positive drift to the right because* $\nabla\log[F/(1-F)] > 0$*. Moreover, the boundary densities and the intensity of the interactions* α*,* β*,* A*,* B *are time-dependent and given by* α^*_t*,* β^*_t*,* A^*_t*,* B^*_t*, respectively.*

Remark 3.2. *As mentioned above, equations* (3.5)–(3.6) *represent the adjoint hydrodynamic equation [that is, the PDE which describes the evolution of the density under the adjoint dynamics]. Hence, in the adjoint dynamics, the density evolves according to a weakly asymmetric exclusion process. The drift at time* t *is* $\log[F_t/(1-F_t)]$*, where* F_t *is the solution to* (1.6) *with* γ *replaced by the density profile at time* t*. The boundary densities and intensities are given by the equations below* (3.9)*.*

Remark 3.3. *Notwithstanding the fact that the boundary densities have been modified and a drift added, a straightforward computation shows that the stationary profile of equation* (3.5) *is still* $\bar\rho$*, the stationary profile of the hydrodynamic equation* (1.1)*. More precisely, as* $F_t = F^{(\gamma)}$ *solves equation* (1.1)*,* $F_t \to \bar\rho$ *as* $t\to\infty$*. Replace in equation* (3.5) R_t *by* $\log[\bar\rho/(1-\bar\rho)]$ *and consider the associated stationary equation [that is, replace* $\partial_t v_t$ *by* 0*, consider this equation in the space variable only and remove the initial condition]. It's easy to check that* $\bar\rho$ *fulfills this stationary equation.*

4. The dynamical rate function

For the reader's convenience, we recall here some properties of the rate functional $I_{[0,T]}(\,\cdot\,|\gamma)$ proved in [19].

The first estimate asserts that the cost of a trajectory in a interval $[0,T]$ is bounded by the sum of its cost in the intervals $[0,S]$ and $[S,T]$.

Let $\tau_r u : \mathbb{R}_+\times[0,1]\to\mathbb{R}$, $r>0$, be the function defined by $\tau_r u(t,x) = u(t+r,x)$. For all $\pi(t,dx) = u(t,x)\,dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ and $0<S<T$,

$$I_{[0,T]}(u\,|\,\gamma) \;\le\; I_{[0,S]}(u\,|\,\gamma) \;+\; I_{[0,T-S]}(\tau_S u\,|\,u(S,\cdot))\;. \tag{4.1}$$

Theorem 4.1. *Fix* $T>0$ *and* $\gamma\in\mathcal{M}_{\rm ac}$*. The function* $I_{[0,T]}(\cdot|\gamma) : D([0,T],\mathcal{M}) \to [0,\infty]$ *is convex, lower semicontinuous and has compact level sets.*

Definition 4.2. *Given $\gamma \in \mathcal{M}_{\rm ac}$, let Π_γ be the collection of all paths $\pi(t,dx) = u(t,x)dx$ in $D([0,T],\mathcal{M}_{\rm ac})$ such that*

(a) *There exists $\mathfrak{t} > 0$, such that u follows the hydrodynamic equation (1.1) in the time interval $[0,\mathfrak{t}]$. In particular, $u(0,\cdot) = \gamma(\cdot)$.*
(b) *For every $0 < \delta \le T$, there exists $\epsilon > 0$ such that $\epsilon \le u(t,x) \le 1-\epsilon$ for all (t,x) in $[\delta,T]\times[0,1]$;*
(c) *u is smooth on $(0,T]\times[0,1]$.*

Theorem 4.3. *Fix $\gamma \in \mathcal{M}_{\rm ac}$. For all π in $D([0,T],\mathcal{M})$ such that $I_{[0,T]}(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in Π_γ such that π^n converges to π in $D([0,T],\mathcal{M})$ and $I_{[0,T]}(\pi^n|\gamma)$ converges to $I_{[0,T]}(\pi|\gamma)$. Moreover, if there exists $\epsilon_0 > 0$ such that $\epsilon_0 \le \gamma \le 1-\epsilon_0$, condition (b) in Definition 4.2 can be replaced by the existence of $\epsilon > 0$ such that $\epsilon \le u(t,x) \le 1-\epsilon$ for all $(t,x) \in [0,T]\times[0,1]$.*

Let Ω_T be the cylinder $(0,T)\times(0,1)$. Fix π in $D([0,T],\mathcal{M}_{\rm ac})$, $\pi(t,dx) = u(t,x)\,dx$. Let $\mathcal{H}^1(\Omega_T)$ be the Hilbert spaces induced by the sets $C^\infty(\Omega_T)$ endowed with the scalar products, $\langle\!\langle G,H\rangle\!\rangle_{1,2}$ defined by

$$\langle\!\langle G,H\rangle\!\rangle_{1,2} \;=\; \int_0^T dt \int_0^1 G_t\, H_t\; dx \;+\; \int_0^T dt\int_0^1 \nabla G_t\, \nabla H_t\; dx\;.$$

Recall from (3.7) the definition of $\mathfrak{p}_{\varrho,D}(a,M)$. For $0<\varrho<1$, $D>0$, $0<a<1$, $M\in\mathbb{R}$, let

$$\mathfrak{c}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{[1-a]\,\varrho\,[1-e^{M}+Me^{M}] \,+\, a\,[1-\varrho]\,[\,1-e^{-M}-Me^{-M}]\,\Big\}\;. \tag{4.2}$$

Note: for a trajectory u_t such that $\delta \le u(t,x)\le 1-\delta$ for all $(t,x)\in[0,T]\times[0,1]$, the space $\mathcal{H}^1(\Omega_T)$ introduced above coincides with the space $\mathcal{H}^1(\sigma(u))$ introduced in [19].

Lemma 4.4. *Fix a trajectory π in $D([0,T],\mathcal{M}_{\rm ac})$, $\pi(t,dx) = u(t,x)\,dx$. Assume that $u\in C^{1,2}([0,T]\times[0,1])$, there exists $\delta>0$ such that $\delta\le u(t,x)\le 1-\delta$ for all $(t,x)\in[0,T]\times[0,1]$ and $I_{[0,T]}(u\,|\,\gamma)$ is finite, where $\gamma = u_0$. Then, there exists a function H in $\mathcal{H}^1(\Omega_T)$ such that u is the unique weak solution to*

$$\begin{cases} \partial_t u \;=\; \Delta u \,-\, 2\,\nabla\{\sigma(u)\;\nabla H\}\;, \\ \nabla u_t(1) \,-\, 2\,\sigma(u_t(1))\,\nabla H_t(1) \;=\; \mathfrak{p}_{\beta,B}\big(\,u_t(1)\,,\,H_t(1)\,\big)\;, \\ \nabla u_t(0) \,-\, 2\,\sigma(u_t(0))\,\nabla H_t(0) \;=\; -\,\mathfrak{p}_{\alpha,A}\big(\,u_t(0)\,,\,H_t(0)\,\big)\;, \\ u(0,\cdot) = \gamma(\cdot)\;. \end{cases} \tag{4.3}$$

Moreover,

$$\begin{aligned} I_{[0,T]}(u\,|\,\gamma) \;&=\; \int_0^T \langle\,\sigma(u_t)\,,\,(\nabla H_t)^2\,\rangle\; dt + \int_0^T \mathfrak{c}_{\beta,B}\big(\,u_t(1)\,,\,H_t(1)\,\big)\; dt \\ &\quad+\; \int_0^T \mathfrak{c}_{\alpha,A}\big(\,u_t(0)\,,\,H_t(0)\,\big)\; dt\;. \end{aligned} \tag{4.4}$$

Weak solutions to equation (4.3) are introduced in Definition B.7. Theorem B.8 states that for each $\gamma\in\mathcal{M}_{\rm ac}$ there exists one and only one weak solution.

5. The Euler-Lagrange equation for S

The Euler–Lagrange equation associated to the variational problem (2.6) is given by the non-linear equation with Robin boundary conditions (1.6). In this section, we provide a precise meaning to this equation, prove existence and uniqueness of solutions, and prove Theorem 2.4. The approach is taken from [4], but there is a serious technical difficulty in the proof of uniqueness. The idea there is to extend the problem to the interval $[-A, 1+B]$, see Lemma 5.5 and the proof of Theorem 5.2.

Recall the definition of $\mathcal{F}$ introduced in (2.5). For $F \in \mathcal{F}$, let

$$\mathcal{R}_\gamma(x) \;=\; \mathcal{R}_\gamma(F\,;\,x) = \big[\,\gamma(x) - F(x)\,\big]\,\frac{\nabla F(x)}{F(x)\,[1-F(x)]}\;, \tag{5.1}$$

With this notation, equation (1.6) takes the form

$$\begin{cases} \Delta F \;=\; \nabla F\,\mathcal{R}_\gamma\;, \\ \nabla F(0) = A^{-1}[F(0)-\alpha]\;, \quad \nabla F(1) = B^{-1}[\beta - F(1)]\;. \end{cases} \tag{5.2}$$

To prove the existence and uniqueness of a solution to (1.6), following [4], we formulate (5.2) as the integro–differential equation

$$F(x) \;=\; \alpha \;+\; (\beta-\alpha)\,\frac{A \;+\; \int_0^x \exp\{\int_0^y \mathcal{R}_\gamma(F;z)\,dz\}\,dy}{A + \int_0^1 \exp\{\int_0^y \mathcal{R}_\gamma(F;z)\,dz\}\,dy + B\,\exp\{\int_0^1 \mathcal{R}_\gamma(F;y)\,dy\}}\;. \tag{5.3}$$

Remark 5.1. *If $\gamma = \bar\rho$, then $F = \bar\rho$ solves (1.6) and (5.3). Moreover, if $F \in C^2([0,1])$ is a solution to the problem (1.6) such that $\nabla F(x) > 0$ for $x \in [0,1]$, then F is also a solution to the integro–differential equation (5.3). Conversely, if $F \in C^1([0,1])$ is a solution to (5.3), then the boundary conditions in (1.6) are satisfied. Moreover, $\nabla F(x) > 0$, $\Delta F(x)$ exists for almost every x and the differential equation in (1.6) holds almost everywhere. Furthermore, if $\gamma \in C([0,1])$, then $F \in C^2([0,1])$ and (1.6) holds everywhere.*

Theorem 5.2. *For each $\gamma \in \mathcal{M}_{\rm ac}$, there exists a unique $F \in \mathcal{F}$ which solves (5.3).*

The existence is proven by applying Schauder's fixed point theorem. The argument requires some notation. For each $\gamma \in \mathcal{M}_{\rm ac}$ consider the map $\mathcal{K}_\gamma : \mathcal{F} \to C^1\big([0,1]\big)$ defined by

$$\mathcal{K}_\gamma(F)(x) \;:=\; \alpha + (\beta-\alpha)\,\frac{A \;+\; \int_0^x \exp\{\int_0^y \mathcal{R}_\gamma(F;z)\,dz\}\,dy}{A + \int_0^1 \exp\{\int_0^y \mathcal{R}_\gamma(F;z)\,dz\}\,dy + B\,\exp\{\int_0^1 \mathcal{R}_\gamma(F;y)\,dy\}}\;. \tag{5.4}$$

Let p and q be given by

$$p \;:=\; \frac{\alpha\,(\beta-\alpha)}{A\alpha + (B+1)\beta}\,\frac{1-\beta}{1-\alpha}\;, \qquad q \;:=\; \frac{(1-\alpha)\,(\beta-\alpha)}{A(1-\alpha) + (B+1)(1-\beta)}\,\frac{\beta}{\alpha}\;. \tag{5.5}$$

Note that $0 < p < q$ because

$$\frac{p}{\beta-\alpha} \;=\; \frac{1}{A + (B+1)(\beta/\alpha)}\,\frac{1-\beta}{1-\alpha} \;<\; \frac{1}{A + (B+1)[(1-\beta)/(1-\alpha)]}\,\frac{\beta}{\alpha} \;=\; \frac{q}{\beta-\alpha}\;.$$

The inequality above follows from the fact that $(1-\beta)\alpha < \beta(1-\alpha)$ as $\alpha < \beta$.

Denote by $\mathcal{B}_{\rm bc}$ the subset of functions in $C^1([0,1])$ which satisfy the boundary conditions of the Euler-Lagrange equation (1.6):

$$\mathcal{B}_{\rm bc} \;:=\; \big\{\, F\in C^1([0,1]) : \nabla F(0) = A^{-1}[F(0)-\alpha]\ ,\ \ \nabla F(1) = B^{-1}[\beta - F(1)]\,\big\}\ ,$$

and by $\mathcal{B}$ the subset of $\mathcal{B}_{\rm bc}$ given by

$$\mathcal{B} \;:=\; \big\{\, F\in \mathcal{B}_{\rm bc} : p\le \nabla F(x) \le q \ \ \forall\, x\in[0,1]\,\big\}\ .$$

Note that $\mathcal{B}_{\rm bc}$, $\mathcal{B}$ are closed and convex, and that $\mathcal{B}$ is contained in $\mathcal{F}$. To establish this last assertion, write that $F(x) \ge F(0) = \alpha + A\nabla F(0) \ge \alpha + Ap > \alpha$ because $\nabla F(x)\ge p>0$. A similar argument shows that $F(x) \le F(1) = \beta - B\nabla F(1) < \beta - Bp < \beta$. In particular, for every $F\in\mathcal{B}$,

$$\alpha \,+\, Ap \;\le\; F(x) \;\le\; \beta \,-\, Bp\ . \tag{5.6}$$

Lemma 5.3. *Fix $\gamma\in\mathcal{M}_{\rm ac}$. Then,*

(a) *The functional $\mathcal{K}_\gamma$ is a continuous map;*
(b) $\mathcal{K}_\gamma(\mathcal{F})\subset\mathcal{B}$;
(c) *There exists a finite constant C_0, such that*

$$\big|\, \nabla \mathcal{K}_\gamma(F)(x) \,-\, \nabla \mathcal{K}_\gamma(F)(y)\,\big| \;\le\; C_0\,|x-y|$$

for all $F\in\mathcal{B}$, x, $y\in[0,1]$.

Proof. Assertion (a) follows from the definitions of $\mathcal{R}_\gamma$ and $\mathcal{K}_\gamma$. We turn to (b). Fix $F\in\mathcal{F}$. It is easy to show that $\mathcal{K}_\gamma(F)$ satisfies the boundary condition of (1.6). It remains to derive the bounds on the derivative of $\mathcal{K}_\gamma(F)$. As $0\le\gamma\le 1$, $-F\le \gamma - F\le 1-F$. Therefore, as $\nabla F\ge 0$ and $\alpha\le F\le\beta$,

$$\frac{-\nabla F}{1-F} \;\le\; \mathcal{R}_\gamma \;\le\; \frac{\nabla F}{F}\ \cdot$$

It follows from these inequalities that

$$\frac{1-\beta}{1-\alpha} \;\le\; \exp\Big\{ \int_0^x \mathcal{R}_\gamma(F;y)\, dy\Big\} \;\le\; \frac{\beta}{\alpha}$$

for all $0\le x\le 1$. Reporting these bounds in the definition of $\mathcal{K}_\gamma(F)$ yields that $p\le \mathcal{K}_\gamma(F)\le q$, as claimed.

The proof of the last assertion of the lemma relies on the previous two bounds and the bound $p\le\nabla F\le q$ which holds for all functions in $\mathcal{B}$. □

Corollary 5.4. *The integro-differential equation* (5.3) *has a solution in $\mathcal{B}$.*

Proof. By Schauder's fixed point theorem, it is enough to show that $\mathcal{K}_\gamma(\mathcal{B})$ has a compact closure in $C^1([0,1])$. By Ascoli–Arzela theorem, this property holds provided $\nabla\mathcal{K}_\gamma(F)$ is Lipschitz continuous, uniformly for $F\in\mathcal{B}$. This is the content of assertion (c) of the lemma. □

We turn to uniqueness. The proof relies on an argument used to prove uniqueness of solutions to equation (1.6) with Dirichlet boundary conditions. First, inspired by [14], we turn the pair (γ, F) defined on the interval $[0,1]$, of solutions to (5.3) [that is, with Robin boundary conditions] into a pair $(\gamma_{\rm ext}, F_{\rm ext})$ defined on the interval $[-A, 1+B]$, of solutions to (5.2) with Dirichlet boundary conditions.

Fix $\gamma\in\mathcal{M}_{\rm ac}$, and let $F\in\mathcal{F}$ be a solution to equation (5.3). We extend γ and F to the interval $[-A,1+B]$ as follows. $F_{\rm ext}$ coincides with F on $[0,1]$, is linear in the complement and $F_{\rm ext}(-A)=\alpha$, $F_{\rm ext}(1+B)=\beta$. $\gamma_{\rm ext}$ coincides with γ on

$[0,1]$ and is equal to $F_{\rm ext}$ in the complement. Hence, $F_{\rm ext}$, $\gamma_{\rm ext} : [-A, 1+B] \to \mathbb{R}$ are given by

$$F_{\rm ext}(x) \;=\; \begin{cases} \alpha \,+\, \nabla F(0)\,[A+x\,] & \text{for } x\in[-A,0)\;, \\ F(x) & \text{for } x\in[0,1]\;, \\ \beta \,+\, \nabla F(1)\,[x-B-1\,] & \text{for } x\in(1,1+B]\;, \end{cases}$$

and

$$\gamma_{\rm ext}(x) \;=\; \begin{cases} \gamma(x) & \text{for } x\in[0,1]\;, \\ F_{\rm ext}(x) & \text{otherwise}\;. \end{cases}$$

Note that $F_{\rm ext}$ belongs to $C^1([-A,1+B])$, $F_{\rm ext}(-A)=\alpha$, $F_{\rm ext}(1+B)=\beta$. Moreover, since $F=\mathcal{K}_\gamma(F)$ and $\mathcal{K}_\gamma(\mathcal{F})\subset\mathcal{B}$, on the interval $[0,1]$, $p\le\nabla F_{\rm ext}(x)\le q$. Hence, $p\wedge\alpha\le\nabla F_{\rm ext}(x)\le q\vee\beta$, and $F_{\rm ext}$ belongs to the set $\mathcal{B}_{\rm ext}$ defined by

$$\begin{aligned} \mathcal{B}_{\rm bc,ext} \;&:=\; \big\{\, G\in C^1([-A,1+B]) : G(-A)=\alpha\;,\; G(1+B)=\beta\,\big\}\;, \\ \mathcal{B}_{\rm ext} \;&:=\; \big\{\, G\in\mathcal{B}_{\rm bc,ext} : p\wedge\alpha\le\nabla F(x)\le q\vee\beta \;\;\forall\, x\in[-A,1+B]\,\big\}\;. \end{aligned}$$

Fix φ in $\mathcal{M}_{\rm ac}([-A,1+B])$. With Dirichlet boundary conditions on the interval $[-A,1+B]$, the problem (5.2) becomes the integro-differential equation

$$G(x) \;=\; \alpha \;+\; (\beta-\alpha)\,\frac{\int_{-A}^{x}\exp\{\int_{-A}^{y}\mathcal{R}_\varphi(G;z)\,dz\}\,dy}{\int_{-A}^{1+B}\exp\{\int_{-A}^{y}\mathcal{R}_\varphi(G;z)\,dz\}\,dy}\;, \tag{5.7}$$

where $\mathcal{R}_\varphi(G;z)$ is given by (5.1).

Lemma 5.5. *Fix $\gamma\in\mathcal{M}_{\rm ac}$, and let $F\in\mathcal{F}$ be a solution to equation* (5.3)*. Then, $F_{\rm ext}$ is a solution to* (5.7) *for $\varphi=\gamma_{\rm ext}$.*

Proof. The assertion follows from a straightforward computation. The result holds for the following reason. On the interval $[0,1]$, the identity (5.2) is in force because F is a solution to (5.3). On the other hand, On the complement, (5.2) holds because both sides of the identity (5.2) vanish. The left-hand side because $F_{\rm ext}$ is linear on $[0,1]^c$, and the right-hand side because $\gamma_{\rm ext}=F_{\rm ext}$. □

Proof of Theorem 5.2. Fix $\gamma\in\mathcal{M}_{\rm ac}$. Existence has been proven in Corollary 5.4. To prove uniqueness, consider two solutions $F^{(1)}$, $F^{(2)}$, and recall the definition of $\gamma^{(j)}_{\rm ext}$, $F^{(j)}_{\rm ext}$, $j=1,\,2$.

By Lemma 5.5, $F^{(1)}_{\rm ext}$, $F^{(2)}_{\rm ext}$ are solutions to (5.7), with $\varphi=\gamma^{(1)}_{\rm ext}$, $\gamma^{(2)}_{\rm ext}$, respectively. Therefore, since these functions solve (5.2) almost everywhere,

$$(\nabla F^{(j)}_{\rm ext})(x) \;=\; (\nabla F^{(j)}_{\rm ext})(-A) \;+\; \int_{-A}^{x}(\nabla F^{(j)}_{\rm ext})(y)\;\mathcal{R}_\gamma(F^{(j)}_{\rm ext};y)\,dy \tag{5.8}$$

for $j=1,\,2$ and all x in $[-A,1+B]$.

Assume that $(\nabla F^{(1)}_{\rm ext})(-A)=(\nabla F^{(2)}_{\rm ext})(-A)$. In this case, by definition of $\gamma^{(j)}_{\rm ext}$, $\gamma^{(1)}_{\rm ext}(x)=\gamma^{(2)}_{\rm ext}(x)$ for all $x\in[-A,0)$. Since this identity always holds for $x\in[0,1]$, $\gamma^{(1)}_{\rm ext}(x)=\gamma^{(2)}_{\rm ext}(x)$ for all $x\in[-A,1]$. By (5.8), elementary bounds and Gronwall's inequality, $(\nabla F^{(1)}_{\rm ext})(x)=(\nabla F^{(2)}_{\rm ext})(x)$ for all $x\in[-A,1]$ so that $F^{(1)}_{\rm ext}(x)=F^{(2)}_{\rm ext}(x)$ for all x in this interval due to the boundary condition satisfied by $F^{(1)}_{\rm ext}$, $F^{(2)}_{\rm ext}$ at $-A$.

Since $(\nabla F^{(1)}_{\rm ext})(1) = (\nabla F^{(2)}_{\rm ext})(1)$, $\gamma^{(1)}_{\rm ext}(x) = \gamma^{(2)}_{\rm ext}(x)$ also for $x \in [1, 1+B]$. The same Gronwall's argument permits to extends the identity $F^{(1)}_{\rm ext}(x) = F^{(2)}_{\rm ext}(x)$ to $x \in [1, 1+B]$. This concludes the argument in the case $(\nabla F^{(1)}_{\rm ext})(-A) = (\nabla F^{(2)}_{\rm ext})(-A)$.

Assume, by contradiction, that $(\nabla F^{(1)}_{\rm ext})(-A) < (\nabla F^{(2)}_{\rm ext})(-A)$. The argument in this case relies on the following identity. As $F^{(1)}_{\rm ext}$ is strictly increasing, (5.2) yields that

$$\nabla \, \frac{F^{(j)}_{\rm ext}\, [1 - F^{(j)}_{\rm ext}]}{\nabla F^{(j)}_{\rm ext}} \;=\; 1 - F^{(j)}_{\rm ext} - \gamma^{(j)}_{\rm ext}$$

holds almost everywhere. Hence, as $F^{(j)}_{\rm ext}(-A) = \alpha$,

$$\frac{F^{(j)}_{\rm ext}(x)\, [1 - F^{(j)}_{\rm ext}(x)]}{(\nabla F^{(j)}_{\rm ext})(x)} \;=\; \frac{\alpha\, [1-\alpha]}{(\nabla F^{(j)}_{\rm ext})(-A)} \;+\; \int_{-A}^{x} [\, 1 - F^{(j)}_{\rm ext}(y) - \gamma^{(j)}_{\rm ext}(y)\,]\, dy \qquad (5.9)$$

for all x in $[-A, 1+B]$.

Let $x_0 := \inf\{y \in (-A, 1+B] : F^{(j)}_{\rm ext}(y) = F^{(2)}_{\rm ext}(y)\}$. The point x_0 belongs to $(-A, 1+B]$ because $F^{(1)}_{\rm ext}(-A) = F^{(2)}_{\rm ext}(-A)$, $(\nabla F^{(1)}_{\rm ext})(-A) < (\nabla F^{(2)}_{\rm ext})(-A)$ and $F^{(1)}_{\rm ext}(1+B) = F^{(2)}_{\rm ext}(1+B)$. Actually, it can not belong to $[-A, 0]$ because the functions $F^{(j)}_{\rm ext}$ are linear in this interval.

Suppose that $x_0 \in (0, 1]$. By definition of x_0, $F^{(1)}_{\rm ext}(x) < F^{(2)}_{\rm ext}(x)$ for all $x \in (-A, x_0)$. Thus, $\gamma^{(1)}_{\rm ext}(x) \le \gamma^{(2)}_{\rm ext}(x)$ for all x in this interval. On the other hand, $F^{(1)}_{\rm ext}(x_0) = F^{(2)}_{\rm ext}(x_0)$ and $(\nabla F^{(1)}_{\rm ext})(x_0) \ge (\nabla F^{(1)}_{\rm ext})(x_0)$. Therefore, by (5.9),

$$\frac{F^{(1)}_{\rm ext}(x_0)[1 - F^{(1)}_{\rm ext}(x_0)]}{(\nabla F^{(1)}_{\rm ext})(x_0)} \;>\; \frac{F^{(2)}_{\rm ext}(x_0)[1 - F^{(2)}_{\rm ext}(x_0)]}{(\nabla F^{(2)}_{\rm ext})(x_0)}$$

or, equivalently, $(\nabla F^{(1)}_{\rm ext})(x_0) < (\nabla F^{(2)}_{\rm ext})(x_0)$, which is a contradiction.

We turn to the case where $x_0 \in (1, 1+B]$. By definition of x_0, $F^{(1)}_{\rm ext}(x) < F^{(2)}_{\rm ext}(x)$ for all $x \in (-A, x_0)$. Since the functions $F^{(j)}_{\rm ext}$ are linear in $[1, 1+B]$, this entails that $x_0 = 1+B$ and that $\gamma^{(1)}_{\rm ext}(x) \le \gamma^{(2)}_{\rm ext}(x)$ for all $x \in [-A, 1+B]$. We may repeat the argument of the previous paragraph to conclude that $(\nabla F^{(1)}_{\rm ext})(1+B) < (\nabla F^{(2)}_{\rm ext})(1+B)$, which is a contradiction. This completes the proof of the theorem. □

Proposition 5.6. *For each $\gamma \in \mathcal{M}_{\rm ac}$, denote by $F = F(\gamma)$ the unique solution in $\mathcal{F}$ of* (5.3)*. Then,*

- (i) *If $\gamma \in C([0,1])$, then $F(\gamma) \in C^2([0,1])$ and it is the unique solution in $\mathcal{F} \cap C^2([0,1])$ of* (5.3)*;*
- (ii) *If γ_n converges to γ in $\mathcal{M}_{\rm ac}$ as $n \to \infty$, then $F_n = F(\gamma_n)$ converges to $F = F(\gamma)$ in $C^1([0,1])$;*

Proof. Existence in assertion (i) follows from Theorem 5.2 and identity (5.3), which holds for all points x in $[0,1]$ because γ is continuous. Uniqueness follows from Theorem 5.2.

To prove (ii), let γ_n be a sequence converging to γ in $\mathcal{M}_{\rm ac}$ and denote by $F_n = F(\gamma_n)$ the corresponding solution to (5.3). By Lemma 5.3.(c) and Ascoli–Arzela theorem, the sequence F_n is relatively compact in $C^1\big([0,1]\big)$. It remains to show uniqueness of its limit points. Consider a subsequence n_j and assume that F_{n_j} converges to G in $C^1\big([0,1]\big)$. Since γ_{n_j} converges to γ in $\mathcal{M}_{\rm ac}$ and F_{n_j} converges to G in $C^1\big([0,1]\big)$, by (5.4) $\mathcal{K}_{\gamma_{n_j}}(F_{n_j})$ converges to $\mathcal{K}_\gamma(G)$. In particular, $G = \lim_j F_{n_j} = \lim_j \mathcal{K}_{\gamma_{n_j}}(F_{n_j}) = \mathcal{K}_\gamma(G)$. Hence, by uniqueness of the solutions to (5.3), $G = F(\gamma)$. This shows that $F(\gamma)$ is the unique limit point of the sequence F_n, and concludes the proof of (ii). □

Fix a trajectory $u(t,\cdot)$, and denote by $F(t,\cdot)$ the function given by $F(t,x) = F(u(t,\cdot))(x)$. In the next lemma, we derive smoothness properties of F in terms of the ones of u. To prove this result, it is convenient to introduce a new variable. Let $\varphi_- := \log[\alpha/(1-\alpha)]$, $\varphi_+ := \log[\beta/(1-\beta)]$ and denote by $\widetilde{\mathcal{F}}$ be the space of monotone C^1 functions given by

$$\widetilde{\mathcal{F}} := \Big\{ \varphi \in C^1([0,1]) \,:\, \varphi_- \,<\, \varphi(x) \,<\, \varphi_+ \,,\ \varphi'(x) > 0 \ \ \forall\, x \in [0,1] \Big\} \,. \tag{5.10}$$

Denote by $\Phi : \mathcal{F} \to \widetilde{\mathcal{F}}$ the map given by

$$\Phi(F) \;=\; \log \frac{F}{1-F} \;\cdot \tag{5.11}$$

Clearly, $\Phi^{-1}(\varphi) = e^\varphi/[1+e^\varphi]$. The advantage of working with $\varphi = \Phi(F)$ instead of F lies in the fact that, as a function of φ, the functional $\mathcal{G}_{\rm bulk}$, defined above (2.6), is concave. This property plays a crucial role in the sequel.

In terms of the variable φ the Euler-Lagrange equation (1.6) becomes

$$\begin{cases} -\,\nabla\Big(\dfrac{1}{\nabla\varphi}\Big) \;+\; \dfrac{1}{1+e^\varphi} \;=\; \gamma \qquad \text{for } \ x \in (0,1) \\ \nabla\varphi(0) \;=\; -\,\dfrac{1}{A}\,\big\{\,(1+e^{-\varphi(0)})\,\alpha \;-\; (1-\alpha)(1+e^{\varphi(0)})\,\big\}\;; \\ \nabla\varphi(1) \;=\; \dfrac{1}{B}\,\big\{\,(1+e^{-\varphi(1)})\,\beta \;-\; (1-\beta)\,(1+e^{\varphi(1)})\,\big\}\;. \end{cases} \tag{5.12}$$

By Lemma 5.3(b), there exists a constant $C_1 = C_1(\alpha,\beta,A,B) \in (0,\infty)$ such that

$$\frac{1}{C_1} \;\le\; (\nabla\varphi)(x) \;\le\; C_1 \qquad \text{for all } x\in[0,1]\,,\, \gamma \in \mathcal{M}_{\rm ac}\,. \tag{5.13}$$

Fix $T>0$ and a trajectory $u(t,\cdot)$, $0\le t\le T$, in $C^{1,0}([0,T]\times[0,1])$ such that $0 \le u(t,x) \le 1$ for all (t,x). Denote by $F(t,x)$ the function given by $F(t,x) = F(u(t,\cdot))(x)$. By Proposition 5.6, F belongs to $C^{0,2}([0,T]\times[0,1])$. Next result asserts that $F\in C^{1,2}([0,T]\times[0,1])$. Let

$$\varphi(t,x) \;:=\; \Phi(F(t,\cdot))(x) \;=\; \log\frac{F(t,x)}{1-F(t,x)} \;\;, \qquad (t,x)\in[0,T]\times[0,1]\,.$$

As F belongs to $C^{0,2}([0,T]\times[0,1])$ and solves (5.2), an elementary computation yields that $\varphi \in C^{0,2}\,([0,T]\times[0,1])$. Moreover, for each $t\in[0,T]$, $\varphi(t)$ is the unique strictly increasing (w.r.t. x) solution to the problem (5.12) with $\gamma = u(t)$. By (5.13), there exists a constant $C_1 = C_1(\alpha,\beta,A,B) \in (0,\infty)$ such that

$$\frac{1}{C_1} \;\le\; (\nabla\varphi)(t,x) \;\le\; C_1 \qquad \forall\,(t,x)\in[0,T]\times[0,1]\,. \tag{5.14}$$

Lemma 5.7. *Fix $u \in C^{1,0}([0,T]\times[0,1])$ and let φ be the corresponding solution to (5.12). Then $\varphi \in C^{1,2}\,([0,T]\times[0,1])$ and for each $0\le t<T$, $\psi := \partial_t\varphi$ is the*

unique classical solution to the linear boundary value problem

$$\begin{cases} \nabla\Big[\dfrac{\nabla\psi}{\big(\nabla\varphi\big)^2}\Big] - \dfrac{e^{\varphi}}{\big(1+e^{\varphi}\big)^2}\,\psi = \partial_t u & x\in(0,1) \\ \nabla\psi \;=\; \dfrac{1}{A}\Big\{(1-\alpha)\,e^{\varphi} \,+\, \alpha\,e^{-\varphi}\Big\}\,\psi & x=0 \\ \nabla\psi \;=\; -\,\dfrac{1}{B}\Big\{(1-\beta)\,e^{\varphi} \,+\, \beta\,e^{-\varphi}\Big\}\,\psi & x=1\,. \end{cases} \tag{5.15}$$

Proof. Fix $t\in[0,T]$. For $h\neq 0$ such that $t+h\in[0,T]$ let $\psi_h(t,x) := h^{-1}\,[\varphi(t+h,x)-\varphi(t,x)]$. By Proposition 5.6, $\psi_h(t,\cdot)\in C^2([0,1])$, and, by (5.12), for $x\in(0,1)$, ψ_h solves

$$\nabla\Big[\,\frac{\nabla\psi_h(t)}{\nabla\varphi(t)\,\nabla\varphi(t+h)}\,\Big] \;-\; \frac{e^{\varphi(t)}}{\big(1+e^{\varphi(t)}\big)\,\big(1+e^{\varphi(t+h)}\big)}\;\frac{e^{h\,\psi_h(t)}-1}{h} \;=\; u_h(t) \tag{5.16}$$

where $u_h(t) = h^{-1}\,[u(t+h)-u(t)]$. At the boundary $x=0$,

$$\nabla\psi_h \;=\; -\,\frac{1}{A}\,\Big\{\,\alpha\,e^{-\varphi(t)}\,\frac{e^{-h\,\psi_h(t)}-1}{h} \;-\; (1-\alpha)\,e^{\varphi(t)}\,\frac{e^{h\,\psi_h(t)}-1}{h}\,\Big\}\;,$$

and at the boundary $x=1$,

$$\nabla\psi_h \;=\; \frac{1}{B}\,\Big\{\,\beta\,e^{-\varphi(t)}\,\frac{e^{-h\,\psi_h(t)}-1}{h} \;-\; (1-\beta)\,e^{\varphi(t)}\,\frac{e^{h\,\psi_h(t)}-1}{h}\,\Big\}\;.$$

Claim 1: The sequence $\psi_h(t)$ is relatively compact in $C([0,1])$.

To prove this claim, multiply equation (5.16) by $-\,\psi_h(t)$ and integrate by parts the first term. Since $\Upsilon(a) := a\,(e^a-1)\geq 0$ for all $a\in\mathbb{R}$ and in view of the expression for $\nabla\psi_h$ at the boundary, the boudary terms resulting from the integration by parts are positive. Therefore,

$$\int_0^1 \frac{\nabla\psi_h(t)^2}{\nabla\varphi(t)\,\nabla\varphi(t+h)}\,dx \;+\; \int_0^1 \frac{e^{\varphi(t)}}{\big(1+e^{\varphi(t)}\big)\,\big(1+e^{\varphi(t+h)}\big)}\;\psi_h(t)\,\frac{e^{h\,\psi_h(t)}-1}{h}\,dx \\ \leq\; -\int_0^1 \psi_h(t)\,u_h(t)\,dx\;.$$

As $\Upsilon(a)\geq 0$ for all $a\in\mathbb{R}$, by (5.14), there exists a finite constant C_0, which depends only on the parameters, such that

$$\int_0^1 \nabla\psi_h(t)^2\,dx \;+\; \frac{1}{h^2}\int_0^1 \Upsilon\big(\,h\,\psi_h(t)\,\big)\,dx \;\leq\; C_0\,\Big|\int_0^1 \psi_h(t)\,u_h(t)\,dx\,\Big|\,. \tag{5.17}$$

On the right-hand side, adding and subtracting $\int_0^1 \psi_h(t)\,dx\int_0^1 u_h(t)\,dx$ inside the absolute value, we estimate this term by

$$C_0\,\|u_h(t)\|_\infty\,\Big\{\,\int_0^1 |\,\nabla\psi_h(t)\,|\,dx \;+\; \int_0^1 \big|\,\psi_h(t)\,\big|\,dx\,\Big\} \tag{5.18}$$

for some finite constant C_0 which depends only on the parameters and may change from line to line. We estimate each term separately. By Young's inequality $2ab\leq Aa^2 + A^{-1}b^2$, $A>0$, the first one is bounded by

$$\frac{1}{2}\int_0^1 |\,\nabla\psi_h(t)\,|^2\,dx \;+\; C_0\,\|u_h(t)\|_\infty^2\;.$$

To bound the second integral in (5.18), let $C_1 = C_0 \|u_h(t)\|_\infty$, and $\delta > 0$ be such that $\Upsilon(a) \ge \delta a^2$ for $|a| \le 1$, and $\Upsilon(a) \ge \delta\,|a|$ for $|a| \ge 1$. Rewrite the second integral as

$$\frac{C_1}{|h|}\Big\{ \int_0^1 \big|\, h\psi_h(t) \,\big|\, \chi_{|\,h\,\psi_h(t)\,|\le 1}\, dx \;+\; \int_0^1 \big|\, h\psi_h(t) \,\big|\, \chi_{|\,h\,\psi_h(t)\,|\ge 1}\, dx \Big\}\,,$$

where $\chi_{\mathcal{A}}$ stands for the indicator of the set $\mathcal{A}$. By Young's inequality and the definition of δ, the previous expression is bounded by

$$\frac{C_1}{|h|}\Big\{ \frac{A}{2} \;+\; \frac{1}{2A}\int_0^1 \big|\, h\psi_h(t) \,\big|^2 \chi_{|\,h\,\psi_h(t)\,|\le 1}\, dx \;+\; \frac{1}{\delta}\int_0^1 \Upsilon(h\psi_h(t))\, dx \Big\}$$

for all $A > 0$. By definition of δ and choosing $A = |h| C_1/\delta$, the previous expression is less than or equal to

$$\frac{C_1^2}{2\delta} \;+\; \frac{1}{h^2}\Big\{\frac{1}{2} \;+\; \frac{C_1\,|h|}{\delta}\Big\}\int_0^1 \Upsilon(h\psi_h(t))\, dx\;.$$

Therefore, (5.18) is bounded above by

$$\frac{1}{2}\int_0^1 |\,\nabla\psi_h(t)\,|^2\, dx \;+\; C_0\,\|u_h(t)\|_\infty^2 \;+\; \frac{C_1^2}{2\delta} \;+\; \frac{1}{h^2}\Big\{\frac{1}{2} + \frac{C_1\,|h|}{\delta}\Big\}\int_0^1 \Upsilon(h\psi_h(t))\, dx\,,$$

where $C_1 = C_0\,\|u_h(t)\|_\infty$.

Reporting this estimate in (5.17) yields that

$$\frac{1}{2}\int_0^1 \nabla\psi_h(t)^2\, dx \;+\; \frac{1}{4h^2}\int_0^1 \Upsilon\big(\, h\,\psi_h(t)\,\big)\, dx \;\le\; C_0\Big(\, 1 \;+\; \frac{1}{2\delta}\Big)\,\|u_h(t)\|_\infty^2 \tag{5.19}$$

for $|h| \le \delta/4C_1$.

This shows that the sequence $\psi_h(t)$ is uniformly Lispchitz continuous, and thus relatively compact in $C([0,1])$, proving the assertion.

Claim 2: The sequence $\psi_h(t)$ converges in $C([0,1])$ to the unique classical solution to (5.15).

Recall from Appendix B the definition of the Sobolev space $\mathcal{H}^1([0,1])$ and of the associated norm. Fix a subsequence $(\psi_{h(k)} : k \ge 1)$, still denoted by ψ_h, which converges to a limit, represented by ψ. By (5.19), ψ belongs to $\mathcal{H}^1([0,1])$ and $\nabla\psi_h(t)$ converges weakly in $\mathcal{L}^2([0,1])$ to $\nabla\psi$.

Fix a function v in $\mathcal{H}^1([0,1])$. Multiply both sides of (5.16) by v and integrate by parts to get that

$$\begin{aligned} &\mathfrak{a}_h(1)\, v(1)\, \nabla\psi_h(t,1) \;-\; \mathfrak{a}_h(0)\, v(0)\, \nabla\psi_h(t,0) \\ &\quad -\int_0^1 \mathfrak{a}_h\, \nabla v\, \nabla\psi_h(t)\, dx + \int_0^1 \mathfrak{b}_h\, v\, \frac{e^{h\,\psi_h(t)} - 1}{h}\, dx \;=\; \int_0^1 u_h(t)\, v\, dx\,, \end{aligned}$$

where $\mathfrak{a}_h = [\,\nabla\varphi(t)\,\nabla\varphi(t+h)\,]^{-1}$, $\mathfrak{b}_h \;=\; -\,e^{\varphi(t)}/(\,1 + e^{\varphi(t)}\,)\,(\,1 + e^{\varphi(t+h)}\,)$. Replace in this equation $\nabla\psi_h(t,0)$, $\nabla\psi_h(t,1)$ by the expressions appearing in the equations below (5.16). As $\mathfrak{a}_h$, $\mathfrak{b}_h$, ψ_h converge in $C([0,1])$, and since $\nabla\psi_h$ converges weakly to $\nabla\psi$ in $\mathcal{L}^2([0,1])$, passing to the limit in the previous equation yields that

$$\begin{aligned} &\mathfrak{a}(1)\, v(1)\, \mathfrak{c}_1\, \psi(t,1) \;-\; \mathfrak{a}(0)\, v(0)\, \mathfrak{c}_0\, \psi(t,0) \\ &\quad -\int_0^1 \mathfrak{a}\, \nabla v\, \nabla\psi(t)\, dx + \int_0^1 \mathfrak{b}\, v\, \psi\, dx \;=\; \int_0^1 (\partial_t u)(t)\, v\, dx\,, \end{aligned}$$

where $\mathfrak{c}_1 = -B^{-1}\{(1-\beta)\, e^{\varphi(t,1)} + \beta\, e^{-\varphi(t,1)}\}$, $\mathfrak{c}_0 = A^{-1}\{(1-\alpha)\, e^{\varphi(t,0)} + \alpha\, e^{-\varphi(t,0)}\}$. Hence, according to [23, IV, Section 1], ψ is a generalized solution to (5.15). By [23, Theorem IV.1.2], the generalized solution is unique, which proves that ψ_h converges in $C([0,1])$ to the unique generalized solution to (5.15). As $\partial_t u(t,\cdot) \in C([0,1])$, by [23, Theorem IV.2.1], the generalized solution belongs to $C^2([0,1])$ and is a classical solution to (5.15). This proves the claim.

It remains to prove the continuity $t \mapsto \psi(t,\cdot)$. According to [23, Theorem IV.1.2], there exists a constant C_0, independent of $\partial_t u$, such that $\|\psi(t)\|_{\mathcal{H}^1([0,1])} \le C_0 \,\|(\partial_t u)(t)\|_{\mathcal{L}^2([0,1])}$. Since there exist a finite constant C_0 such that $\|v\|_\infty \le C_0\|v\|_{\mathcal{H}^1([0,1])}$ for all $v \in \mathcal{H}^1([0,1])$,

$$\|\, \psi(t+h) - \psi(t)\, \|_\infty \;\le\; C_0 \, \big\|\, (\partial_t u)(t+h) \;-\; (\partial_t u)(t)\, \big\|_{\mathcal{L}^2([0,1])} \;.$$

This proves that ψ belongs to $C^{0,2}([0,T]\times[0,1])$, and therefore that φ belongs to $C^{1,2}([0,T]\times[0,1])$, as claimed. □

Proof of Theorem 2.4. For each $F \in \mathcal{F}$, $\mathcal{G}(\cdot, F)$ is a convex, lower semi-continuous functional on $\mathcal{M}_{\rm ac}$. The functional $S_0(\cdot)$ inherits these properties. By choosing $F = \bar\rho$ in (2.6) we obtain that for every $\gamma \in \mathcal{M}_{\rm ac}$,

$$\mathcal{G}(\gamma, \bar\rho) \;=\; S_{\rm eq}(\gamma) \;-\; (1+A+B)\, \log(1+A+B)\;,$$

where $S_{\rm eq} \colon \mathcal{M}_{\rm ac} \to \mathbb{R}$ is the convex and nonnegative functional

$$S_{\rm eq}(\gamma) \;=\; \int_0^1 \Big\{\, \gamma(x)\, \log \frac{\gamma(x)}{\bar\rho(x)} + \big[1-\gamma(x)\big] \log \frac{1-\gamma(x)}{1-\bar\rho(x)} \,\Big\}\, dx\;.$$

As $S_{\rm eq}$ is non-negative, $S_0(\gamma) \ge -(1+A+B)\, \log(1+A+B)$. On the other hand, as $a \mapsto \log a$ is concave, by Jensen's inequality and since $\alpha \le F(x) \le \beta$, for every $F \in \mathcal{F}$, $\gamma \in \mathcal{M}_{\rm ac}$,

$$\mathcal{G}_{\rm bulk}(\gamma, F) \;\le\; \log \frac{1}{\alpha} \;+\; \log \frac{1}{1-\beta}\;\cdot$$

Hence, there exists a finite constant $C_0 = C_0(\alpha,\beta,A,B)$ such that $S_0(\gamma) \le C_0$ for all $\gamma \in \mathcal{M}_{\rm ac}$. This proves the first assertion of the theorem. We turn to the second.

Recall from (5.11) the definition of $\varphi = \Phi(F) \in \widetilde{\mathcal{F}}$, and that $F = \Phi^{-1}(\varphi) = e^\varphi/[1+e^\varphi]$. Set $\widetilde{\mathcal{G}}_{\rm bulk}(\gamma,\varphi) = \mathcal{G}_{\rm bulk}(\gamma, \Phi^{-1}(\varphi))$ so that

$$\widetilde{\mathcal{G}}_{\rm bulk}(\gamma,\varphi) \;=\; \int_0^1 \Big\{ \mathfrak{h}(\gamma) \;+\; (1-\gamma)\,\varphi \;-\; \log\big[\, 1+e^{\varphi}\,\big] \;+\; \log \frac{\varphi'}{\beta-\alpha} \Big\}\, dx\;, \tag{5.20}$$

where $\mathfrak{h}(a) = a\log a \;+\; (1-a)\log(1-a)$.

To prove the second assertion of the theorem, we have to show that for each $\gamma \in \mathcal{M}_{\rm ac}$, the supremum over the set $\widetilde{\mathcal{F}}$ of

$$\widetilde{\mathcal{G}}(\gamma,\varphi) \;:=\; \widetilde{\mathcal{G}}_{\rm bulk}(\gamma,\varphi) \;+\; A\, \ln \frac{F(0)-\alpha}{A(\beta-\alpha)} \;+\; B\, \ln \frac{\beta - F(1)}{B(\beta-\alpha)}$$

is uniquely attained at $\varphi = \Phi(F(\gamma))$, where, recall, $F(\gamma)$ represents the unique solution to (5.3). In the previous equation, $F(x)$ stands for $\exp\{\varphi(x)\}/[1+\exp\{\varphi(x)\}]$, $x = 0,\, 1$.

Since the functions $a \mapsto \log a$, $a \mapsto -\log(1+e^a)$, $a \mapsto -\log\{\, [e^a/(1+e^a)] - \alpha\}$, $a \mapsto -\log\{\, \beta - [e^a/(1+e^a)]\,\}$ are strictly concave, the last two in the interval (φ_-, φ_+) defined above (5.10), for each $\gamma \in \mathcal{M}_{\rm ac}$, the functional $\widetilde{\mathcal{G}}(\gamma,\cdot)$ is strictly concave on $\widetilde{\mathcal{F}}$. Moreover it is easy to show that $\widetilde{\mathcal{G}}(\gamma,\cdot)$ is Gateaux differentiable on $\widetilde{\mathcal{F}}$ with derivative given by

$$\Big\langle \frac{\delta\widetilde{\mathcal{G}}(\gamma,\varphi)}{\delta\varphi}\,,\,g\Big\rangle \;=\; \int_0^1 \Big\{\,\frac{g'}{\varphi'} \,+\, \Big[\,\frac{1}{1+e^{\varphi}}\,-\,\gamma\,\Big]\,g\,\Big\}\,dx$$
$$+\;\frac{A\,g(0)}{(1-\alpha)(1+e^{\varphi(0)})-\alpha(1+e^{-\varphi(0)})}\;+\;\frac{B\,g(1)}{(1-\beta)(1+e^{\varphi(1)})-\beta(1+e^{-\varphi(1)})}$$

for all g in $C^1([0,1])$. By (5.12), the right-hand side vanishes for $\varphi=\Phi(F(\gamma))$.

By [16, Proposition 1.5.4] and since $\widetilde{\mathcal{G}}(\gamma,\cdot)$ is strictly concave, for any $\psi\neq\varphi$ in $\widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{G}}(\gamma,\psi)\;<\;\widetilde{\mathcal{G}}(\gamma,\varphi)\;+\;\Big\langle\,\frac{\delta\widetilde{\mathcal{G}}(\gamma,\varphi)}{\delta\varphi}\,,\,\psi-\varphi\,\Big\rangle\,.$$

Since $\delta\widetilde{\mathcal{G}}(\gamma,\varphi)/\delta\varphi=0$ for $\varphi=\Phi(F(\gamma))$, the supremum on $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{G}}(\gamma,\cdot)$ is uniquely attained when $\varphi=\Phi(F(\gamma))$. □

Remark 5.8. *Fix $\gamma\in\mathcal{M}_{\rm ac}$, and consider a sequence $\gamma_n\in\mathcal{M}_{\rm ac}$ such that*

(i) *For each $n\geq 1$, there exists $\delta_n>0$ such that $0<\delta_n\leq\gamma_n(x)\leq 1-\delta_n$ for all $x\in[0,1]$;*

(ii) *γ_n converges to γ a.e.*

Then, by the dominated convergence theorem and Proposition 5.6.(ii),

$$\lim_{n\to\infty}S_0(\gamma_n)\;=\;\lim_{n\to\infty}\mathcal{G}\big(\gamma_n,F(\gamma_n)\big)\;=\;\mathcal{G}\big(\gamma,F(\gamma)\big)\;=\;S_0(\gamma)\,.$$

6. The quasi-potential

Let $\delta_0>0$ be such that $\delta_0\leq\alpha<\beta\leq 1-\delta_0$. For $\delta\in(0,\delta_0]$ and $T>0$, let

$$\begin{aligned}\mathcal{M}_\delta \;&:=\;\big\{\gamma\in C^2([0,1])\,:\,\delta\leq\gamma(x)\leq 1-\delta\,\big\}\\ D_{T,\delta}\;&:=\;\big\{\,u\in C^{1,2}([0,T]\times[0,1])\,:\,\delta\leq u(t,x)\leq 1-\delta\,\big\}\,.\end{aligned}\tag{6.1}$$

Unless otherwise stated, throughout this section, $T>0$ and $0<\delta\leq\delta_0$ are fixed.

Lemma 6.1. *Fix u in $D_{T,\delta}$ and denote by $F(t,x)=F(u(t,\cdot))\,(x)$ the solution to the boundary value problem* (1.6) *with γ replaced by $u(t)$. Set*

$$\Gamma(t,x)\;=\;\log\frac{u(t,x)}{1-u(t,x)}\;-\;\log\frac{F(t,x)}{1-F(t,x)}\,.\tag{6.2}$$

Then, for each $T\geq 0$,

$$S_0\big(u(T)\big)-S_0\big(u(0)\big)\;=\;\int_0^T\langle\,\partial_t u(t)\,,\,\Gamma(t)\,\rangle\,dt\,.\tag{6.3}$$

Proof. Recall that $F(t,\cdot)$ is strictly increasing for any $t\in[0,T]$. By Lemma 5.7, F belongs to $C^{1,2}\big([0,T]\times[0,1]\big)$. By Theorem 2.4 and the dominated convergence theorem,

$$\begin{aligned}\frac{d}{dt}S_0(\,u(t)\,)\;&=\;\frac{d}{dt}\,\mathcal{G}(\,u(t),F(t)\,)\\ &=\;\big\langle\,\partial_t u(t)\,,\,\Gamma(t)\,\big\rangle\;+\;\big\langle\,\partial_t F(t),\,\frac{F(t)-u(t)}{F(t)[1-F(t)]}\,\big\rangle\;+\;\big\langle\,\frac{1}{\nabla F(t)}\,,\,\partial_t\nabla F(t)\,\big\rangle\\ &\quad+\;A\,\frac{(\partial_t F)(t,0)}{F(t,0)-\alpha}\;+\;B\,\frac{(\partial_t F)(t,1)}{F(t,1)-\beta}\,.\end{aligned}$$

As F belongs to $\mathcal{B}$ it satisfies mixed boundary conditions at $x = 0$, $x = 1$. An integration by parts yields that the previous expression is equal to

$$\big\langle\, \partial_t u(t)\,,\, \Gamma(t)\,\big\rangle \;+\; \Big\langle\, \partial_t F(t), \frac{F(t) - u(t)}{F(t)[1-F(t)]} \;+\; \frac{\Delta F(t)}{\big(\nabla F(t)\big)^2}\,\Big\rangle\,.$$

To conclude the proof, it remains to recall Remark 5.1, which asserts that F solves (1.6) almost everywhere. □

Recall the definition of the Hamiltonian $\mathcal{H}$, given in (1.2), and the one of $\mathcal{M}_\delta$, introduced at the beginning of this section.

Lemma 6.2. *Fix $\gamma \in \mathcal{M}_\delta$, and let $F = F(\gamma)$ be the solution of the boundary value problem* (1.6)*. Set*

$$\Gamma(x) \;=\; \log\frac{\gamma(x)}{1-\gamma(x)} \;-\; \log\frac{F(x)}{1-F(x)}\,\cdot$$

Then,

$$\mathcal{H}(\,\gamma\,,\,\Gamma\,) \;=\; 0\,.$$

Proof. By Corollary 5.4 and Proposition 5.6, $F \in \mathcal{M}_\delta$. By definition of Γ and since F belongs to $\mathcal{B}$,

$$\begin{aligned} & \mathfrak{b}_{\alpha,A}\big(\,\gamma(0)\,,\,\Gamma(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\,\gamma(1)\,,\,\Gamma(1)\,\big) \\ &\quad =\; [\,F(0)-\gamma(0)\,]\,\frac{(\nabla F)(0)}{F(0)\,[1-F(0)]} \;-\; [\,F(1)-\gamma(1)\,]\,\frac{(\nabla F)(1)}{F(1)\,[1-F(1)]}\,\cdot \end{aligned} \tag{6.4}$$

On the other hand, a straightforward computation yields that

$$\begin{aligned} & \big\langle \gamma(1-\gamma)\,,\,\big(\nabla\Gamma\big)^2\big\rangle \;-\; \big\langle \nabla\gamma\,,\,\nabla\Gamma\big\rangle \\ &\quad =\; \Big\langle \gamma(1-\gamma)\,,\,\Big(\frac{\nabla F}{F(1-F)}\Big)^2\Big\rangle \;-\; \Big\langle \nabla\gamma\,,\,\frac{\nabla F}{F(1-F)}\Big\rangle\,. \end{aligned}$$

Rewrite the second term as

$$-\,\Big\langle \nabla(\gamma - F)\,,\,\frac{\nabla F}{F(1-F)}\Big\rangle \;-\; \Big\langle \nabla F\,,\,\frac{\nabla F}{F(1-F)}\Big\rangle\,,$$

and integrate by parts the first expression. The boundary terms cancel with the ones appearing in (6.4).

Up to this point, we proved that

$$\mathcal{H}(\,\gamma\,,\,\Gamma\,) \;=\; \Big\langle\,\gamma(1{-}\gamma)\,,\,\Big(\frac{\nabla F}{F(1-F)}\Big)^2\,\Big\rangle + \Big\langle\gamma{-}F\,,\,\nabla\frac{\nabla F}{F(1-F)}\Big\rangle - \Big\langle\nabla F\,,\,\frac{\nabla F}{F(1-F)}\Big\rangle\,.$$

Since $\gamma(1-\gamma) - F(1-F) = (\gamma - F)(1-\gamma-F)$, we may rewrite this sum as

$$\Big\langle\,\gamma - F\,,\,(1-\gamma-F)\Big(\frac{\nabla F}{F(1-F)}\Big)^2\,\Big\rangle \;+\; \Big\langle\,\gamma - F\,,\,\nabla\frac{\nabla F}{F(1-F)}\Big\rangle\,.$$

On the other hand, as $\nabla\{\nabla F/F(1{-}F)\} = \Delta F/F(1{-}F) - (1{-}2F)[\nabla F/F(1{-}F)]^2$, the previous expression is equal to

$$\Big\langle\,\gamma - F\,,\,(F-\gamma)\Big(\frac{\nabla F}{F(1-F)}\Big)^2\,\Big\rangle \;+\; \Big\langle\,\gamma - F\,,\,\frac{\Delta F}{F(1-F)}\Big\rangle\,.$$

This sum vanishes because F is the solution to (1.6). □

Remark 6.3. *Lemma 6.1 identifies* Γ *as the functional derivative of* S*,*

$$\Gamma \;=\; \frac{\delta S_0}{\delta \gamma} \;=\; \frac{\delta S}{\delta \gamma}\;,$$

and Lemma 6.2 states that this derivative $\Gamma = \delta S/\delta\gamma$ *satisfies the Hamilton–Jacobi equation.*

6.1. **Lower bound for the quasi-potential.** In this subsection, we prove that $V \ge S$.

Lemma 6.4. *For each* $\gamma \in \mathcal{M}_{\rm ac}$*,* $V(\gamma) \ge S(\gamma)$*.*

Proof. In view of the variational definition of V, we have to show that $S(\gamma) \le I_{[0,T]}(u|\bar\rho)$ for any $T>0$ and any path $u \in D\big([0,T];\mathcal{M}\big)$ which connects the stationary profile $\bar\rho$ to γ in the time interval $[0,T]$: $u(0)=\bar\rho$, $u(T)=\gamma$.

Fix such a path u and assume first that u belongs to $D_{T,\delta}$ for some $\delta>0$. For $0\le t\le T$, let $F(t) = F(u(t))$ be the solution to the elliptic problem (1.6) with $u(t)$ in place of γ. In view of the variational definition of $I_{[0,T]}(u|\bar\rho)$ given in (2.3), to prove that $S(\gamma) \le I_{[0,T]}(u|\bar\rho)$ it is enough to exhibit some function $H \in C^{1,2}([0,T]\times[0,1])$ for which $S(\gamma) \le J_{T,H}(u)$. We claim that Γ given in (6.2) fulfills these conditions.

Note that Γ belongs to $C^{1,2}([0,T]\times[0,1])$ because, on the one hand, u belongs to this set as it is assumed to be in $D_{T,\delta}$. On the other hand, by Lemma 5.7, $F \in C^{1,2}([0,T]\times[0,1])$.

Recall the definition of the Hamiltonian $\mathcal{H}$ introduced in (1.2). By (2.2), integrating by parts in time yields that

$$J_{T,\Gamma}\big(u\big) \;=\; \int_0^T \Big\{ \big\langle \partial_t u(t)\,,\, \Gamma(t) \big\rangle \,-\, \mathcal{H}\big(\,u(t)\,,\,\Gamma(t)\,\big) \Big\}\, dt\;.$$

By Lemmata 6.1 and 6.2, $J_{T,\Gamma}(u) = S_0(u(T)) \,-\, S_0(u(0)) = S_0(\gamma) - S_0(\bar\rho) = S(\gamma)$.

Up to this point we have shown that $S(\gamma) \le I_{[0,T]}(u|\bar\rho)$ for smooth paths u bounded away from 0 and 1. We extend this result to arbitrary paths

Fix a path u with finite rate function: $I_{[0,T]}(u|\bar\rho) < \infty$. Since $\alpha \le \bar\rho \le \beta$, by Theorem 4.3, there exists a sequence $\{u^n,\, n\ge 1\}$, $u^n \in D_{T,\delta_n}$ for some $\delta_n>0$, such that u_n converges to u and $I_{[0,T]}(u_n|\bar\rho)$ converges to $I_{[0,T]}(u|\bar\rho)$. Therefore, by the result on smooth paths and the lower semi continuity of S,

$$I_{[0,T]}(u|\bar\rho) \;=\; \lim_{n\to\infty} I_{[0,T]}(u_n|\bar\rho) \ge \liminf_{n\to\infty} S\big(u_n(T)\big) \ge S(u(T))\;,$$

which concludes the proof of the lemma. □

6.2. **Upper bound, the adjoint hydrodynamic equation.** The following lemma explains which is the right candidate for the optimal path for the variational problem (2.4). For $0<\varrho<1$, $D>0$, $0<a<1$, $M\in\mathbb{R}$, let

$$\mathfrak{q}_{\varrho,D}(a,M) \;=\; \frac{1}{D}\Big\{ [1-a]\,\varrho\,\big[\,e^{M} - M - 1\,\big] \,+\, a\,[1-\varrho]\,\big[\,e^{-M} + M - 1\,\big] \Big\}\,. \tag{6.5}$$

Note that $\mathfrak{q}_{\varrho,D}(a,\cdot\,)$ is a nonnegative, convex function which vanishes at the origin. Recall the definition of $\mathfrak{p}_{\varrho,D}(a,M)$, introduced in (4.2).

Lemma 6.5. *Fix a profile* $\psi \in \mathcal{M}_\delta$*, and a path* $u \in D_{T,\delta}$ *with finite rate function,* $I_{[0,T]}(u|\psi) < \infty$*. For* $0 \le t \le T$*, denote by* $F(t) = F(u(t))$ *the unique solution* *to the boundary value problem* (1.6) *with* ϕ *replaced by* $u(t)$. *Then, there exists a function* $K \in \mathcal{H}^1(\Omega_T)$ *such that* u *is the weak solution to*

$$\begin{cases} \partial_t u \;=\; -\,\Delta u \;+\; 2\,\nabla\Big(\sigma(u)\,\nabla\big[\log\dfrac{F}{1-F} + K\big]\Big)\;, \quad (t,x)\in[0,T]\times(0,1) \\ \nabla u_t(1) \;-\; 2\,\sigma(u_t(1))\,\nabla G_t(1) \;=\; \mathfrak{p}_{\beta,B}\big(\,u_t(1)\,,\,G_t(1)\,\big)\;, \\ \nabla u_t(0) \;-\; 2\,\sigma(u_t(0))\,\nabla G_t(0) \;=\; -\,\mathfrak{p}_{\alpha,A}\big(\,u_t(0)\,,\,G_t(0)\,\big)\;, \\ u(0,x) \;=\; \psi(x)\;, \quad x\in[0,1]\;. \end{cases} \tag{6.6}$$

where $G_t = \Gamma_t - K_t$ *and* Γ_t *is given by* (6.2). *Moreover,*

$$\begin{aligned} I_{[0,T]}(u|\psi) \;&=\; S_0(u(T)) \;-\; S_0(\psi) \;+\; \int_0^T \big\langle\,\sigma(u(t))\,,\,[\nabla K(t)]^2\,\big\rangle \\ &+\; \int_0^T e^{G_t(1)}\,\mathfrak{q}_{\beta,B}(u_t(1),K_t(1))\,dt \;+\; \int_0^T e^{G_t(0)}\,\mathfrak{q}_{\alpha,A}(u_t(0),K_t(0))\,dt\;. \end{aligned} \tag{6.7}$$

Strategy of the proof of the upper bound: We present below the main steps of the proof in light of Lemma 6.5. Fix a density profile $\gamma\in\mathcal{M}_{\rm ac}$ and denote by $(v^{(\gamma)},F^{(\gamma)})$ the solution to the time-reversed equation (6.6) with $K=0$ and starting from γ:

$$\begin{cases} \partial_t v \;=\; \Delta v \;-\; 2\,\nabla\big(\,\sigma(v)\,\nabla\log\dfrac{F}{1-F}\,\big) \quad (t,x)\in(0,\infty)\times(0,1)\;, \\ \nabla v_t(1) \;-\; 2\,\sigma(v_t(1))\,\nabla H_t(1) \;=\; \mathfrak{p}_{\beta,B}\big(\,v_t(1)\,,\,H_t(1)\,\big)\;, \\ \nabla v_t(0) \;-\; 2\,\sigma(v_t(0))\,\nabla H_t(0) \;=\; -\,\mathfrak{p}_{\alpha,A}\big(\,v_t(0)\,,\,H_t(0)\,\big)\;, \\ v(0,\cdot) = \gamma(\,\cdot\,)\;, \quad x\in[0,1]\;, \end{cases} \tag{6.8}$$

$$\begin{cases} \Delta F_t = \big(v_t - F_t\big)\dfrac{\big(\nabla F_t\big)^2}{F_t(1-F_t)} \quad (t,x)\in(0,\infty)\times(0,1)\;, \\ \nabla F_t(0) = A^{-1}[F_t(0)-\alpha]\;, \quad \nabla F_t(1) = B^{-1}[\beta - F_t(1)]\;. \end{cases} \tag{6.9}$$

In equation (6.8), $H_t = \log[v_t/(1-v_t)] - \log[F_t/(1-F_t)]$ and $\mathfrak{p}_{\varrho,D}$ has been introduced in (4.2). Equation (6.8) will be shown to be equivalent to (3.5). Proposition 6.7 provides a precise meaning to the coupled equation (6.8)–(6.9).

The second step consists in proving that the solution $v_t^{(\gamma)}$ of equation (6.8) converges to $\bar\rho$ as $t\to\infty$. This is the content of Lemma 6.10, where we prove that this convergence takes place in $\mathcal{L}^\infty$.

Fix T_1 large enough for $v^{(\gamma)}(T_1)$ to be close to $\bar\rho$ in $\mathcal{L}^\infty$. Reverse in time the path $v^{(\gamma)}$ by setting $w^{(1)}(t) = v^{(\gamma)}(T_1 - t)$, $0\le t\le T_1$. The path $w^{(1)}$ satisfies equation (6.6) with $K=0$ and $\psi = v^{(\gamma)}(T_1)$. Therefore, by Lemma 6.5,

$$I_{[0,T_1]}(\,w^{(1)}\,|\,v^{(\gamma)}(T_1)\,) \;=\; S_0(\gamma) \;-\; S_0(v^{(\gamma)}(T_1))\;.$$

It remains to replace in the previous formula $v^{(\gamma)}(T_1)$ by $\bar\rho$, keeping in mind that $v^{(\gamma)}(T_1)$ is close to $\bar\rho$ in the $\mathcal{L}^\infty$ norm. This is done in Lemma 6.11, where we show that if ϕ is close to $\bar\rho$ in $\mathcal{L}^\infty$, then there exists a path $w_t^{(2)}$, $0\le t\le 1$, which connects $\bar\rho$ to ϕ and such that $I_{[0,1]}(\,w^{(2)}\,|\,\bar\rho\,) \le C_0\,\|\phi-\bar\rho\|_2^2$.

Define the path $w(t)$, $0\le t\le T_1+1$, by $w(t) = w^{(2)}(t)$ for $0\le t\le 1$, $w(t) = w^{(1)}(t-1)$, $1\le t\le T_1+1$. By definition $w(0)=\bar\rho$ and $w(T_1+1)=\gamma$. Moreover, by the previous bounds of the rate functional, and since $w(1) = v^{(\gamma)}(T_1)$,

$$\begin{aligned} I_{[0,T_1+1]}(\, w \,|\, \bar\rho\,) \; &= \; I_{[0,1]}(\, w^{(2)} \,|\, \bar\rho\,) \; + \; I_{[1,T_1+1]}(\, w^{(1)}(1+\,\cdot\,) \,|\, v^{(\gamma)}(T_1)\,) \\ &\le \; S_0(\gamma) \; - \; S_0(v^{(\gamma)}(T_1)) \; + \; C_0 \, \|v^{(\gamma)}(T_1) - \bar\rho\|_2^2 \; . \end{aligned}$$

The first identity says that the cost of a path in the time-interval $[0, T_1+1]$ is equal to its cost in the interval $[0,1]$ plus its cost in the interval $[1, T_1+1]$. Equation (4.1) states that the inequality holds, which is enough for the argument. By lower semicontinuity of S_0, and since $v^{(\gamma)}(T_1) \to \bar\rho$ as $T_1 \to \infty$, $S_0(\bar\rho) \le \liminf_{T_1\to\infty} S_0(v^{(\gamma)}(T_1))$. Hence, for all $\epsilon > 0$, there exists T_1 large enough such that

$$I_{[0,T_1+1]}(\, w \,|\, \bar\rho\,) \; \le \; S_0(\gamma) \; - \; S_0(\bar\rho) \; + \; \epsilon \; .$$

This proves that $V(\gamma) \le S(\gamma)$, as claimed.

Proof of Lemma 6.5. Denote by H the function in $\mathcal{H}^1(\Omega_T)$ introduced in Lemma 4.4, and recall from (6.2) the definition of Γ. Set $K := \Gamma - H$, so that $G = H$. The function K belongs to $\mathcal{H}^1(\Omega_T)$ because, by hypothesis, $u \in D_{T,\delta}$ and, by Lemma 5.7, $F \in C^{1,2}(\,[0,T]\times[0,1]\,)$. Then (6.6) follows easily from (4.3).

We turn to the identity (6.7), Note that $\partial_t u \; = \; \Delta u \; - \; 2\,\nabla(\,\sigma(u)\,\nabla\,[\,\Gamma - K\,]\,)$. In (6.3), replace $\partial_t u(t)$ by the right-hand side of this identity and integrate by parts to get that

$$\begin{aligned} & S_0(u(T)) \; - \; S_0(\psi) \; = \; - \int_0^T \big\langle \nabla u(t) \,,\, \nabla \Gamma(t) \big\rangle \, dt \\ & \quad + \; 2 \int_0^T \big\langle\, \sigma(u(t))\, \nabla(\, \Gamma(t) - K(t)\,) \,,\, \nabla\Gamma(t) \,\big\rangle \, dt \\ & \quad + \int_0^T \mathfrak{p}_{\beta,B}\big(\, u_t(1) \,,\, H_t(1)\,\big)\, \Gamma_t(1)\, dt \; + \; \int_0^T \mathfrak{p}_{\alpha,A}\big(\, u_t(0) \,,\, H_t(0)\,\big)\, \Gamma_t(0)\; dt \; . \end{aligned}$$

By Lemma 6.2, the previous expression is equal to

$$\begin{aligned} & - \int_0^T \big\langle\, \sigma(u(t))\,(\, \nabla\Gamma(t)\,)^2 \,\big\rangle \; dt \; + \; 2 \int_0^T \big\langle\, \sigma(u(t))\, \nabla(\, \Gamma(t) - K(t)\,) \,,\, \nabla\Gamma(t) \,\big\rangle \; dt \\ & - \int_0^T \mathfrak{b}_{\beta,B}\big(\, u_t(1) \,,\, \Gamma_t(1)\,\big) \; dt \; - \; \int_0^T \mathfrak{b}_{\alpha,A}\big(\, u_t(0) \,,\, \Gamma_t(0)\,\big) \; dt \\ & + \int_0^T \mathfrak{p}_{\beta,B}\big(\, u_t(1) \,,\, H_t(1)\,\big)\, \Gamma_t(1) \; dt \; + \; \int_0^T \mathfrak{p}_{\alpha,A}\big(\, u_t(0) \,,\, H_t(0)\,\big)\, \Gamma_t(0) \; dt \; . \end{aligned}$$

Up to this point, we expressed the difference $S_0(u(T)) \; - \; S_0(\psi)$ as a sum of many terms. Add on both sides of this identity $\int_0^T \big\langle\, \sigma(u(t))\,(\, \nabla K(t)\,)^2 \,\big\rangle \; dt$. Add and subtract on the right-hand side

$$\int_0^T \mathfrak{c}_{\beta,B}\big(\, u_t(1) \,,\, H_t(1)\,\big) \; dt \; + \; \int_0^T \mathfrak{c}_{\alpha,A}\big(\, u_t(0) \,,\, H_t(0)\,\big) \; dt \; .$$

Recall that $H = \Gamma - K = G$ and, from (4.4), the identity satisfied by $I_{[0,T]}(u|\psi)$, to get after these summations that

$$\begin{aligned} & S_0(u(T)) \; - \; S_0(\psi) \; + \; \int_0^T \big\langle\, \sigma(u(t))\,(\, \nabla K(t)\,)^2 \,\big\rangle \; dt \\ & \; = \; I_{[0,T]}(u|\psi) \; - \; \int_0^T e^{H_t(1)}\, \mathfrak{q}_{\beta,B}(u_t(1), K_t(1)) \; dt \; - \; \int_0^T e^{H_t(0)}\, \mathfrak{q}_{\alpha,A}(u_t(0), K_t(0)) \; dt \; , \end{aligned}$$

as claimed [because $H = G$]. □

We turn to the proof of the upper bound for the quasi-potential, as described in the Strategy of the proof. We first simplify the boundary conditions in equations (6.8)–(6.9).

As $H_t = \log[v_t/(1-v_t)] - \log[F_t/(1-F_t)]$, the boundary terms are given by

$$\begin{aligned} &-\nabla v_t \,+\, 2\,\frac{\sigma(v_t)}{\sigma(F_t)}\,\nabla F_t \;=\; \frac{1}{B\,\sigma(F_t)}\,\big\{\,\beta\, v_t\,[\,1-F_t\,]^2 \,-\, (1-\beta)\,(1-v_t)\,F_t^2\,\big\}\;, \\ &-\nabla v_t \,+\, 2\,\frac{\sigma(v_t)}{\sigma(F_t)}\,\nabla F_t \;=\; \frac{-\,1}{A\,\sigma(F_t)}\,\big\{\,\alpha\, v_t\,[\,1-F_t\,]^2 \,-\, (1-\alpha)\,(1-v_t)\,F_t^2\,\big\}\;, \end{aligned} \tag{6.10}$$

for $x = 1,\, 0$, respectively. Let $R = \log[F/(1-F)]$. With this notation, equation (6.8) can be written as

$$\begin{cases} \partial_t v \;=\; \Delta v \,-\, 2\,\nabla\big(\,\sigma(v)\,\nabla R\,\big) \quad (t,x)\in(0,\infty)\times(0,1)\;, \\ \nabla v_t(1) \,-\, 2\,\sigma(v_t(1))\,\nabla R_t(1) \;=\; \mathfrak{p}_{1-\beta,B}\big(\,v_t(1)\,,\, R_t(1)\,\big)\;, \\ \nabla v_t(0) \,-\, 2\,\sigma(v_t(0))\,\nabla R_t(0) \;=\; -\,\mathfrak{p}_{1-\alpha,A}\big(\,v_t(0)\,,\, R_t(0)\,\big)\;, \\ v(0,\cdot) = \gamma\;, \quad x\in[0,1]\;, \end{cases} \tag{6.11}$$

where F_t is the solution to (6.9). Note that this equation corresponds to equation (3.5).

Proposition 6.7 provides a precise meaning for the system of equations (6.8)–(6.9), or, equivalently, (6.11)–(6.9). This proposition requires an estimate on the solutions to equation (1.1). Denote by $u^{(\gamma)}$, $\gamma\in\mathcal{M}_{\rm ac}$, the solution to (1.1) with initial conditions γ, and by $F = F(\gamma)$ the solution to (1.6). Recall the definition of the constant p and q introduced in (5.5).

Lemma 6.6. *For every $\gamma\in\mathcal{M}_{\rm ac}$, $(t,x)\in\mathbb{R}_+\times[0,1]$, $\alpha + Ap \,\le\, u^{(F(\gamma))}(t,x) \,\le\, \beta - Bp$. Moreover, for every $T>0$, there exists a constant $c_1 = c_1(A,B,\alpha,\beta,T)>0$ such that $c_1 \le \nabla u^{(F(\gamma))}(t,x) \le c_1^{-1}$ for all $\gamma\in\mathcal{M}_{\rm ac}$, $(t,x)\in[0,T]\times[0,1]$.*

Proof. By (5.6) and Corollary 5.4, $\alpha + Ap \,\le\, F(\gamma)(x) \,\le\, \beta - Bp$ for all $\gamma\in\mathcal{M}_{\rm ac}$, $x\in[0,1]$. The first assertion of the lemma follows from Theorem B.4.

By Corollary 5.4, $F = F(\gamma)$ belongs to $\mathcal{B}$. Therefore, $p \le \nabla F(x) \le q$ for all $0\le x\le 1$, $\gamma\in\mathcal{M}_{\rm ac}$. Let $v = \nabla u^{(F(\gamma))}$. Then, v solves the equation

$$\begin{cases} \partial_t v \;=\; \Delta v \\ v(t,0) \;=\; A^{-1}\,[\,u^{(F(\gamma))}(t,0) - \alpha\,] \\ v(t,1) \;=\; B^{-1}\,[\beta \,-\, u^{(F(\gamma))}(t,1)\,] \\ v(0,\cdot) \;=\; \nabla F(\cdot)\;. \end{cases} \tag{6.12}$$

The maximum principle, Theorem 2 of [24, Chapter 3], states that the maximum and the minimum of v are attained at the boundary. The assertion of the lemma follows from the bounds on $u^{(F(\gamma))}$ and ∇F obtained above. These estimates are uniform over $\gamma\in\mathcal{M}_{\rm ac}$. □

Proposition 6.7. *Fix $\gamma\in\mathcal{M}_{\rm ac}$, and denote by $F^{(\gamma)}(t) = u^{(F(\gamma))}(t)$ the solution to the heat equation (1.1) with initial condition $F^{(\gamma)}(0,\cdot) = F(\gamma)(\cdot)$. Define $v^{(\gamma)} = v^{(\gamma)}(t,x)$ by (3.8). Then, $v^{(\gamma)}(0,\cdot) = \gamma(\cdot)$, $v^{(\gamma)}$ is smooth in $(0,\infty)\times[0,1]$ and $(v^{(\gamma)}, F^{(\gamma)})$ satisfies (6.9), (6.11) in $(0,\infty)\times[0,1]$.*

Proof. Fix $\gamma \in \mathcal{M}_{\rm ac}$, and let $F(\gamma)$ be the solution of (1.6). By Corollary 5.4, $F(\gamma)$ belongs to $C^1([0,1])$ and there is a constant $c_0 \in (0,\infty)$, depending only on the parameters, such that $c_0 \le [\nabla F(\gamma)]\,(x) \le c_0^{-1}$ for all $x \in [0,1]$.

Let $F^{(\gamma)}(t)$ be the solution to (1.1) with initial condition $F^{(\gamma)}(0) = F(\gamma)$. By Theorem B.4, $F^{(\gamma)}$ is smooth in $(0,\infty) \times [0,1]$. By Lemma 6.6, for every $T > 0$, there exists $c_1 = c_1(A, B, \alpha, \beta, T) \in (0,\infty)$ such that $c_1 \le (\nabla F^{(\gamma)})(t,x) \le c_1^{-1}$ for all $(t,x) \in [0,T] \times [0,1]$.

Define $v^{(\gamma)}_t$ by equation (3.8) which we reproduce here:

$$v^{(\gamma)}_t \;=\; F^{(\gamma)}_t \;+\; \sigma(\, F^{(\gamma)}_t \,)\, \frac{\Delta F^{(\gamma)}_t}{(\nabla F^{(\gamma)}_t)^2}\;, \quad t \,\ge\, 0\;. \tag{6.13}$$

For $t = 0$, we may replace on the right-hand side $F^{(\gamma)}_0$ by $F(\gamma)$ to get that $v^{(\gamma)}_0 = \gamma$, as claimed.

In view of the regularity and the bounds obtained for $F^{(\gamma)}$ in the previous paragraph, $v^{(\gamma)}$ is smooth in $(0,\infty) \times [0,1]$. Moreover, by (6.13), the pair $(v^{(\gamma)}, F^{(\gamma)})$ satisfies (6.9).

It remains to show that $v^{(\gamma)}$ fullfils (6.11).

Claim 1: The function $v^{(\gamma)}_t$ complies with the boundary conditions of (6.11).

We prove this assertion for $x = 1$, the other one being similar. By (6.13), at the boundary, as $\nabla F^{(\gamma)}_t(1) = [\beta - F^{(\gamma)}_t(1)]/B$, taking a time-derivative on both sides of the identity yields that $(\nabla^3 F^{(\gamma)}_t)(1) = -\,(1/B)\,(\Delta F^{(\gamma)}_t)(1)$ because $\partial_t F^{(\gamma)} = \Delta F^{(\gamma)}$.

Compute, separately, the right and left-hand sides of the right boundary condition in (6.11). We start with the left-hand side of the identity. Since $(\nabla^3 F^{(\gamma)}_t)(1) = -(1/B)(\Delta F^{(\gamma)}_t)(1)$, taking a space derivative in (6.13) yields that, at $x = 1$,

$$-\,\nabla v^{(\gamma)} \;=\; -\,\nabla F^{(\gamma)} - (1{-}2F^{(\gamma)})\,\frac{\Delta F^{(\gamma)}}{\nabla F^{(\gamma)}} + \frac{\sigma(F^{(\gamma)})}{B}\,\frac{\Delta F^{(\gamma)}}{(\nabla F^{(\gamma)})^2} + 2\,\sigma(F^{(\gamma)})\,\frac{(\Delta F^{(\gamma)})^2}{(\nabla F^{(\gamma)})^3}\;.$$

On the other hand, by (6.13) and a straightforward computation,

$$2\,\frac{\sigma(v^{(\gamma)})}{\sigma(F^{(\gamma)})}\,\nabla F^{(\gamma)} \;=\; 2\,\nabla F^{(\gamma)}\Big\{\,1 \,+\, (1 - 2F^{(\gamma)})\,\frac{\Delta F^{(\gamma)}}{(\nabla F^{(\gamma)})^2} \,-\, \sigma(F^{(\gamma)})\,\frac{(\Delta F^{(\gamma)})^2}{(\nabla F^{(\gamma)})^4}\,\Big\}\;.$$

Summing the previous two identities yields that, at $x = 1$,

$$-\,\nabla v^{(\gamma)} \;+\; 2\,\frac{\sigma(v^{(\gamma)})}{\sigma(F^{(\gamma)})}\,\nabla F^{(\gamma)} \;=\; \nabla F^{(\gamma)} \;+\; \{\,\beta \,-\, 2\beta F^{(\gamma)} \,+\, (F^{(\gamma)})^2\,\}\,\frac{\Delta F^{(\gamma)}}{B\,(\nabla F^{(\gamma)})^2}\;.$$

A simple calculation gives that the right-hand side of the right boundary condition in (6.11) is also equal to this quantity. This proves Claim 1.

Claim 2: The function $v^{(\gamma)}_t$ fullfils (6.11) *in the interior.*

The proof of this claim is identical to the one presented in [3, Appendix B] and in [4, Lemma 5.5]. We reproduce it here in sake of completeness.

From (6.13),

$$\frac{v^{(\gamma)}(1 - v^{(\gamma)})}{F^{(\gamma)}(1 - F^{(\gamma)})} = 1 + (1 - 2F^{(\gamma)})\frac{\Delta F^{(\gamma)}}{\big(\nabla F^{(\gamma)}\big)^2} - F^{(\gamma)}(1 - F^{(\gamma)})\frac{\big(\Delta F^{(\gamma)}\big)^2}{\big(\nabla F^{(\gamma)}\big)^4}\;.$$

As $F^{(\gamma)}$ solves the heat equation (1.1), a long computation yields that

$$\big(\partial_t - \Delta\big)\Big(\sigma(F^{(\gamma)})\,\frac{\Delta F^{(\gamma)}}{\big(\nabla F^{(\gamma)}\big)^2}\Big) \;=\; -\,2\,\nabla\Big(\frac{\sigma(v^{(\gamma)})}{\sigma(F^{(\gamma)})}\,\nabla F^{(\gamma)}\Big)\,.$$

By (6.13), $v^{(\gamma)}$ satisfies the differential equation in (6.11), as claimed. □

Lemma 6.8. *Under the hypotheses of Proposition 6.7, assume that γ belongs to $C^2([0,1])$. Then, $v^{(\gamma)}$ belongs to $C^{1,2}([0,\infty)\times[0,1])\cap C([0,\infty);\mathcal{M}_{\rm ac})$.*

Proof. Assume that γ belongs to $C^2([0,1])$. By Remark 5.1 and (1.6), $\Delta F(\gamma)$ belongs to $C^2([0,1])$. Taking time derivatives in (1.1) yields that $\Delta F^{(\gamma)}(t)$ is the solution to (B.2) with initial condition $\Delta F(\gamma)$. By Theorem B.2 (a), $(t,x)\mapsto \Delta F^{(\gamma)}(t,x)$ belongs to $C^{1,2}([0,\infty)\times[0,1])$. Therefore, by (6.13), $v^{(\gamma)}$ belongs to $C^{1,2}([0,\infty)\times[0,1])$

Claim 1: The function $v^{(\gamma)}_t$ belongs to $C(\mathbb{R}_+,\mathcal{M}_{\rm ac})$.

We have to show that $0\le v^{(\gamma)}(t,x)\le 1$ for all (t,x). In view of (3.9), equation (6.11) describes the macroscopic evolution of the density for weakly asymmetric boundary driven exclusion processes with weak boundary interaction. The drift is given by ∇R_t where $R_t=\log[F^{(\gamma)}_t/(1-F^{(\gamma)}_t)]$. By the first part of the proof and Lemma 6.6, R_t belongs to $C^{1,2}([0,\infty)\times[0,1])$.

As $v^{(\gamma)}_0=\gamma$ and $0\le\gamma\le 1$, by the hydrodynamic limit of theses systems, derived in [19], this equation has a weak solution taking values in the interval $[0,1]$, cf. Theorem B.8. Since $v^{(\gamma)}$ belongs to $C^{1,2}([0,\infty)\times[0,1])$ and solves (6.11) pointwisely, it is a weak solution to equation (6.11) in the sense of Definition B.7. Therefore, by the uniqueness of weak solutions of (6.11), Theorem B.8, $v^{(\gamma)}$ coincides with solution obtained in the proof of the hydrodynamic limit which takes values in $[0,1]$. This completes the proof of the lemma. □

Lemma 6.9. *Under the hypotheses of Proposition 6.7, assume that $\delta\le\gamma(x)\le 1-\delta$ a.e. for some $\delta>0$. Then, there exists $\delta'=\delta'(A,B,\alpha,\beta,\delta)\in(0,1)$, such that $\delta'\le v^{(\gamma)}(t,x)\le 1-\delta'$ for all $(t,x)\in(0,\infty)\times[0,1]$.*

Proof. The proof is divided in several assertions. Fix $t>0$.

Claim 1: If $v^{(\gamma)}(t,\cdot)$ has a local maximum at $x_0\in(0,1)$ and $v^{(\gamma)}(t,x_0)>1-\alpha$, then $\partial_t v^{(\gamma)}(t,x_0)<0$.

Assume that $v^{(\gamma)}(t,\cdot)$ has a local maximum at $x_0\in(0,1)$. Since $v^{(\gamma)}$ is a smooth solution to (6.8), $(\nabla v^{(\gamma)})(t,x_0)=0$. By (6.13) and a straightforward computation, $\Delta\log\{F^{(\gamma)}/(1-F^{(\gamma)})\}=(v^{(\gamma)}+F^{(\gamma)}-1)\,(\nabla F^{(\gamma)})^2/\sigma(F^{(\gamma)})^2$. Therefore, by (6.11), at the point (t,x_0),

$$\partial_t v^{(\gamma)} \;=\; \Delta v^{(\gamma)} \;-\; 2\,\sigma(v^{(\gamma)})\,(v^{(\gamma)}+F^{(\gamma)}-1)\,\frac{(\nabla F^{(\gamma)})^2}{\sigma(F^{(\gamma)})^2}\,.$$

As x_0 is a local maximum, $\Delta v^{(\gamma)}\le 0$. On the other hand, since $v^{(\gamma)}(t,x_0)>1-\alpha$, and, by Lemma 6.6, $\alpha\le F^{(\gamma)}$, $v^{(\gamma)}+F^{(\gamma)}-1>0$ so that $\partial_t v^{(\gamma)}<0$, which proves the claim.

The same argument shows that $(\partial_t v^{(\gamma)})(t,x_1)>0$ if $x_1\in(0,1)$ is a minimum of $v^{(\gamma)}(t,\cdot)$ and $v^{(\gamma)}(t,x_1)<1-\beta$.

We turn to the possibility that the maximum is attained at the boundary. By Lemma 6.6, $F^{(\gamma)}$ takes value in the interval $[\alpha + pA, \beta - pB]$. Let

$$\nu \;:=\; \max_{\alpha+pA\le\varphi\le\beta-pB} \frac{(1-\beta)\varphi^2}{\beta - 2\beta\varphi + \varphi^2} \;<\; 1\;.$$

As $\nu < 1$, there exists $a > 0$ such that $\nu/(1-2a) < 1$ and $aBp < 1$.

Claim 2: If $v^{(\gamma)}_t(1) \,>\, \nu_r \,:=\, \max\{\nu/(1-2a)\,,\, 1 - aBp\}$, then $\nabla v^{(\gamma)}_t(1) < 0$.

Since $v^{(\gamma)}$ solves equation (6.11), in view of the definition of R, the first equation in (6.10) holds with $v^{(\gamma)}$, $F^{(\gamma)}$ in place of v, F, respectively. At the boundary, $\nabla F^{(\gamma)}_t(1) = (1/B)[\beta - F^{(\gamma)}_t(1)]$. Therefore,

$$\begin{aligned}\nabla v^{(\gamma)}_t \;&=\; \frac{1}{B\,\sigma(F^{(\gamma)}_t)}\Big\{2\,\sigma(v^{(\gamma)}_t)\,[\beta - F^{(\gamma)}_t] \,+\, (1-\beta)\,(1-v^{(\gamma)}_t)\,[F^{(\gamma)}_t]^2 \,-\, \beta\, v^{(\gamma)}_t\,[\,1 - F^{(\gamma)}_t\,]^2\Big\}\\ &=\; \frac{-1}{B\,\sigma(F^{(\gamma)}_t)}\Big\{\, v^{(\gamma)}_t \Big[\,\{\,\beta - 2\beta F^{(\gamma)}_t + [F^{(\gamma)}_t]^2\,\} \,-\, 2\,(1-v^{(\gamma)}_t)\,(\beta - F^{(\gamma)}_t)\Big] \,-\, (1-\beta)\,[F^{(\gamma)}_t]^2\,\Big\}\;.\end{aligned}$$

In these formulas, we wrote $v^{(\gamma)}_t$, $F^{(\gamma)}_t$ for $v^{(\gamma)}_t(1)$, $F^{(\gamma)}_t(1)$, respectively. As $v^{(\gamma)}_t(1) > 1 - a\,B\,p$ and $Bp \le \beta - F^{(\gamma)}_t$, the last term is bounded by

$$\begin{aligned}&\frac{-1}{B\,\sigma(F^{(\gamma)}_t)}\Big\{\, v^{(\gamma)}_t \Big[\,\{\,\beta - 2\beta F^{(\gamma)}_t + [F^{(\gamma)}_t]^2\,\} \,-\, 2\,a\,(\beta - F^{(\gamma)}_t)^2\Big] \,-\, (1-\beta)\,[F^{(\gamma)}_t]^2\,\Big\}\\ &\quad\le\; \frac{-1}{B\,\sigma(F^{(\gamma)}_t)}\Big\{\,(1-2a)\,v^{(\gamma)}_t\,\{\,\beta - 2\beta F^{(\gamma)}_t + [F^{(\gamma)}_t]^2\,\} \,-\, (1-\beta)\,[F^{(\gamma)}_t]^2\,\Big\}\;.\end{aligned}$$

This expression is negative by definition of ν and because we assumed that $v^{(\gamma)}_t(1) > \nu/(1-2a)$. This proves Claim 2.

Let

$$\mu_r \;:=\; \min_{\alpha+pA\le\varphi\le\beta-pB} \frac{(1-\beta)\varphi^2}{\beta - 2\beta\varphi + \varphi^2} \;>\; 0\;.$$

Claim 3: If $v^{(\gamma)}_t(1) \,<\, \mu_r$, then $\nabla v^{(\gamma)}_t(1) > 0$.

Recall the formula for $\nabla v^{(\gamma)}_t(1)$ presented at the beginning of the proof of Claim 2. Since $2(1 - v^{(\gamma)}_t)\,(\beta - F^{(\gamma)}_t) \,>\, 0$ and $v^{(\gamma)}_t(1) \,<\, \mu_r$,

$$\nabla v^{(\gamma)}_t \;>\; \frac{1}{B\,\sigma(F^{(\gamma)}_t)}\Big\{\,(1-\beta)\,[F^{(\gamma)}_t]^2 \,-\, \mu_r\,\{\,\beta - 2\beta F^{(\gamma)}_t + [F^{(\gamma)}_t]^2\,\}\,\Big\}\;.$$

As $F^{(\gamma)}_t$ takes values in the interval $[\alpha + pA, \beta - pB]$, the right-hand side is non-negative by definition of μ_r. This proves Claim 3.

Similarly, one can be prove the existence of $\mu_l \,>\, 0$ and $\nu_l \,<\, 1$ such that $\nabla v^{(\gamma)}_t(0) > 0$ if $v^{(\gamma)}_t(0) \,>\, \nu_l$; $\nabla v^{(\gamma)}_t(0) < 0$ if $v^{(\gamma)}_t(0) \,<\, \mu_l$.

This result together with Claims 1, 2, 3, yield that $\min\{\mu_l\,,\, \mu_r\,,\, \delta\,,\, 1-\beta\,\} \,\le\, v^{(\gamma)}(t,x) \,\le\, \max\{\nu_l\,,\, \nu_r\,,\, 1-\delta\,,\, 1-\alpha\,\}$ for all $(t,x) \in \mathbb{R}_+ \times [0,1]$, which concludes the proof of the lemma. □

Next result states that the solution to (6.8), as constructed in Proposition 6.7, converges to $\bar\rho$, as $t \to \infty$, uniformly with respect to the initial condition γ.

Lemma 6.10. *Let $v^{(\gamma)}$, $\gamma \in \mathcal{M}_{\rm ac}$, be given by* (6.13)*. Then,*

$$\lim_{t\to\infty} \sup_{\gamma\in\mathcal{M}_{\rm ac}} \big\|\, v^{(\gamma)}(t) - \bar\rho\,\big\|_\infty \;=\; 0\;.$$

Proof. Write the solution $F^{(\gamma)}(t)$ of (1.1) as $F^{(\gamma)}(t,x) = \bar\rho(x) + \Psi^{(\gamma)}(t,x)$. Then, $\Psi^{(\gamma)}$ solves the equation (B.2) with $\phi = F(\gamma) - \bar\rho$. In particular, by Theorem B.2, $\Psi^{(\gamma)}(t)$ can be represented as $\Psi^{(\gamma)}(t) = P_t^{(R)}\Psi^{(\gamma)}(0)$. Since $\Psi^{(\gamma)}(0) = F(\gamma) - \bar\rho$ and the solution $F(\gamma)$ of (1.6) as well as $\bar\rho$ are contained in the interval $[\alpha,\beta]$, we have that $\|\Psi^{(\gamma)}(0)\|_\infty \le \beta - \alpha < 1$, uniformly over $\gamma \in \mathcal{M}_{\rm ac}$. Therefore, expressing $\Psi^{(\gamma)}(t) = P_t^{(R)}\Psi^{(\gamma)}(0)$ in the basis $(f_j : j \ge 1)$ by standard arguments (see [26, Corollary 2 of Section 4.4]),

$$\lim_{t\to\infty} \sup_{\gamma\in\mathcal{M}_{\rm ac}} \|\Psi^{(\gamma)}(t)\|_\infty \;=\; 0\;.$$

Taking a time derivative at the boundary yields that $\Delta\Psi^{(\gamma)}(t)$ also solves equation (B.2). By (1.6), as $F(\gamma)$ belongs to $\mathcal{B}$, $\|\Delta\Psi^{(\gamma)}(0)\|_\infty \le C_0(A,B,\alpha,\beta)$. Thus, as $\Delta\bar\rho = 0$, by the same argument,

$$\lim_{t\to\infty} \sup_{\gamma\in\mathcal{M}_{\rm ac}} \|\Delta\Psi^{(\gamma)}(t)\|_\infty \;=\; 0\;.$$

Expressing the derivative $(\nabla\Psi^{(\gamma)})(t,x)$ as $(\nabla\Psi^{(\gamma)})(t,0)+\int_0^x(\Delta\Psi^{(\gamma)})(t,y)\,dy$, and since $(\nabla\Psi^{(\gamma)})(t,0) = A^{-1}\,\Psi^{(\gamma)}(t,0)$, we deduce from the two previous results that

$$\lim_{t\to\infty} \sup_{\gamma\in\mathcal{M}_{\rm ac}} \|(\nabla\Psi^{(\gamma)})(t)\|_\infty \;=\; 0\;.$$

By (6.13), the assertion of the lemma follows from the previous estimates. □

Lemma 6.10 shows that we may join a profile γ in $\mathcal{M}_{\rm ac}$ to a neighborhood of the stationary profile by using the equation (6.8) for a time interval $[0,T_1]$ which at the same time regularizes the profile. As described in the Strategy of the proof, it remains to connect $v^{(\gamma)}(T_1)$ to $\bar\rho$. In the next lemma we show that this can be done by paying only a small price. Denote by $\|\cdot\|_2$ the norm in $\mathcal{L}^2([0,1])$, and recall the definition of δ_0, given at the beginning of this section, and of the set $D_{T,\delta}$, introduced in (6.1). In the lemma below, λ_1 represents the smallest eigenvalue of the Robin Laplacian (cf. Appendix A).

Lemma 6.11. *Let $\gamma \in \mathcal{M}_{\rm ac}$ be a smooth profile such that $\|\gamma-\bar\rho\|_\infty \le \delta_0\,\min\{(1/4), (1/\Lambda)\}$, where $\Lambda = 16\sqrt{A/\lambda_1}$. Then, there exist a smooth path $w(t)$, $t\in[0,1]$, with $\delta_0/2 \le w(t) \le 1-\delta_0/2$, $w(0) = \bar\rho$, $w(1) = \gamma$ and a finite constant $C_0 = C_0(\delta_0)$ such that*

$$I_{[0,1]}(w|\bar\rho) \;\le\; C_0\,\|\gamma-\bar\rho\|_2^2\;.$$

In particular, for profiles γ satisfying the hypotheses of this lemma, $V(\gamma) \le C_0\,\|\gamma-\bar\rho\|_2^2$.

The "straight path" $w(t) = \bar\rho\,(1-t)+\gamma\,t$ yields a bound in terms of the $\mathcal{H}_1([0,1])$ norm of $\gamma-\bar\rho$. In contrast, the path below, similar to the one proposed in [4], provides a bound in terms of the $\mathcal{L}^2$ norm. We assume in the proof below that the reader is familiar with the notation and results presented in Appendix A.

Proof. Recall that we denote by $\{f_j : j\ge 1\}$ the orthonormal eigenfunctions of the Robin Laplacian and by λ_j the associated eigenvalues.

We claim that the path $w(t) = w(t,x)$, $(t,x)\in[0,1]\times[0,1]$ given by

$$w(t) \;=\; \bar\rho \;+\; \sum_{k\ge1} \frac{e^{\lambda_k t}-1}{e^{\lambda_k}-1}\,\langle\gamma-\bar\rho\,,\,f_k\rangle\,f_k \tag{6.14}$$

fulfills the conditions stated in the lemma. By [26, Corollary 2 of Section 4.4], this sum is absolutely convergent, uniformly in $x \in [0,1]$.

Clearly, $w(0) = \bar\rho$, $w(1) = \gamma$ and w satisfies the boundary conditions

$$(\nabla w)(t,0) \;=\; A^{-1}\,[w(t,0) - \alpha]\;, \quad (\nabla w)(t,1) \;=\; B^{-1}[\beta - w(t,1)]\;. \tag{6.15}$$

By the smoothness assumption on γ, $w \in C^{1,2}([0,1]\times[0,1])$.

In order to show that $\delta_0/2 \le w \le 1-\delta_0/2$, write $w(t)$ as $w(t) = \bar\rho + q(-t)$. Clearly, $q(t) = q(t,x)$, $(t,x) \in [-1,0]\times[0,1]$ solves the equation

$$\begin{cases} \partial_t q(t) \;=\; \Delta q(t) \;-\; g \\ (\nabla q)(t,0) \;=\; A^{-1}\, q(t,0)\;, \\ (\nabla q)(t,1) \;=\; -\,B^{-1} q(t,1)\;, \\ q(-1) = \gamma - \bar\rho\;, \end{cases} \tag{6.16}$$

where $g = g(x)$ is given by

$$g \;=\; \sum_{k\ge 1} \frac{\lambda_k}{e^{\lambda_k}-1} \langle \gamma - \bar\rho\,,\, f_k\rangle\, f_k\;.$$

Recall the definition of the $\mathcal{H}_R$-norm $\|g\|_{\mathcal{H}_R}$ induced by the Robin Laplacian. By definition of g and since $a^2 \le 2(e^a-1)$, $a>0$, $\|\gamma-\bar\rho\|_\infty \le \delta_0/\Lambda$,

$$\begin{aligned} \|g\|^2_{\mathcal{H}_R} \;&=\; \sum_{k\ge1} \lambda_k \Big(\frac{\lambda_k}{e^{\lambda_k}-1}\Big)^2 \langle \gamma-\bar\rho, f_k\rangle^2 \;\le\; \frac{4}{\lambda_1} \sum_{k\ge 1} \langle \gamma-\bar\rho, f_k\rangle^2 \\ &=\; \frac{4}{\lambda_1}\, \|\gamma-\bar\rho\|_2^2 \;\le\; \frac{4}{\lambda_1}\frac{\delta_0^2}{\Lambda^2}\;. \end{aligned}$$

The solution $q(t)$ of (6.16) can be expressed as $P^{(R)}_{t+1}(\gamma-\bar\rho) + \int_{-1}^t P^{(R)}_{t-s} g\, ds$. Therefore, since, by hypothesis, $\|\,\gamma-\bar\rho\,\|_\infty \le \delta_0/4$, by (B.4), (A.11) and the previous bound,

$$\sup_{t\in[-1,0]} \|q(t)\|_\infty \;\le\; \|\,\gamma-\bar\rho\,\|_\infty \;+\; \|\,g\,\|_\infty \;\le\; \frac{\delta_0}{4} \;+\; \sqrt{2(A\vee 1)}\,\frac{2}{\sqrt{\lambda_1}}\,\frac{\delta_0}{\Lambda}\;.$$

In particular, by definition of Λ, w belongs to $D_{1,\delta_0/2}$.

We turn to the cost of the path w. Since w is a smooth path such that $w(0)=\bar\rho$, in formula (2.1), integrate by parts twice in space and once in time to get that

$$\begin{aligned} J_{1,H}(w) \;&=\; \int_0^1 \big\langle \partial_t w_t \;-\; \Delta w_t\,,\, H_t\big\rangle\, dt \;-\; \int_0^1 \big\langle \sigma(w_t), \big(\nabla H_t\big)^2\big\rangle\, dt \\ &-\; \int_0^1 \nabla w_t(1)\, H_t(1)\; dt \;+\; \int_0^1 \nabla w_t(0)\, H_t(0)\; dt \\ &-\; \int_0^1 \Big\{ \mathfrak{b}_{\alpha,A}\big(\, w_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{b}_{\beta,B}\big(\, w_t(1)\,,\, H_t(1)\,\big) \Big\}\, dt\;. \end{aligned}$$

for all H in $C^{1,2}([0,1]\times[0,1])$. Recall the definition of $\mathfrak{q}_{\varrho,D}(a,M)$, introduced in (6.5). As w satisfies the boundary conditions (6.15), we may rewrite the previous identity as

$$J_{1,H}(w) \;=\; \int_0^1 \big\langle \partial_t w_t \,-\, \Delta w_t\,,\, H_t \big\rangle\, dt \;-\; \int_0^1 \big\langle \sigma(w_t), \big(\nabla H_t\big)^2 \big\rangle\, dt \\ -\; \int_0^1 \Big\{ \mathfrak{q}_{\alpha,A}\big(\, w_t(0)\,,\, H_t(0)\,\big) \;+\; \mathfrak{q}_{\beta,B}\big(\, w_t(1)\,,\, H_t(1)\,\big) \Big\}\, dt\;.$$

As $\delta_0/2 \le w \le 1-\delta_0/2$, $\sigma(w_t) \ge (\delta_0/2)^2$. On the other hand, for $\delta_0/2 \le a, \varrho \le 1-\delta_0/2$, $\mathfrak{q}_{\varrho,D}(a,M) \ge (2/D)\,(\delta_0/2)^2\,[\cosh M - 1] \ge (1/D)\,(\delta_0/2)^2\, M^2$. Therefore,

$$J_{1,H}(w) \;\le\; \int_0^1 \big\langle \partial_t w_t \,-\, \Delta w_t\,,\, H_t \big\rangle\, dt \;-\; \Big(\frac{\delta_0}{2}\Big)^2 \int_0^1 \big\langle \big(\nabla H_t\big)^2 \big\rangle\, dt \\ -\; \Big(\frac{\delta_0}{2}\Big)^2 \int_0^1 \Big\{ \frac{1}{A} H_t(0)^2 \;+\; \frac{1}{B} H_t(1)^2 \Big\}\, dt\;.$$

By (A.7), we may rewrite this inequality as

$$J_{1,H}(w) \;\le\; \int_0^1 \big\langle \partial_t w_t \,-\, \Delta w_t\,,\, H_t \big\rangle\, dt \;-\; \Big(\frac{\delta_0}{2}\Big)^2 \int_0^1 \big\|\, H_t \,\big\|^2_{\mathcal{H}_R}\, dt\;.$$

Since $Q_{[0,1]}(w) < \infty$, the rate functional $I_{[0,1]}(w|\bar\rho)$ is given by the variational formula (2.3). Therefore, maximizing over H on both sides of the previous displayed equation yields that

$$I_{[0,1]}(w|\bar\rho) \;\le\; \int_0^1 \Big(\frac{2}{\delta_0}\Big)^2 \sup_{G\in C^2([0,1])} \Big\{ \big\langle h(t)\,,\, G \big\rangle \;-\; \big\|\, G \,\big\|^2_{\mathcal{H}_R} \Big\}\, dt\;, \tag{6.17}$$

where $h(t) = \partial_t w_t \,-\, \Delta w_t$.

By (6.14),

$$h(t) \;=\; \sum_{k\ge 1} \lambda_k\, \frac{2e^{\lambda_k t} - 1}{e^{\lambda_k} - 1}\, \langle \gamma - \bar\rho\,,\, f_k \rangle\, f_k$$

Hence, by Young's inequality and (A.10), for all $G \in C^2([0,1])$,

$$\langle h(t)\,,\, G \rangle \;=\; \sum_{k\ge 1} \langle h(t)\,,\, f_k \rangle\, \langle G\,,\, f_k \rangle \;\le\; \sum_{k\ge 1} \frac{1}{4\,\lambda_k} \langle h(t)\,,\, f_k \rangle^2 \;+\; \big\|\, G \,\big\|^2_{\mathcal{H}_R}\;.$$

By the formula for $h(t)$, the last sum is equal to

$$\frac{1}{4} \sum_{k\ge 1} \lambda_k \Big(\frac{2e^{\lambda_k t} - 1}{e^{\lambda_k} - 1}\Big)^2 \langle \gamma - \bar\rho\,,\, f_k \rangle^2 \;\le\; J^2 \sum_{k\ge 1} \lambda_k\, e^{2\lambda_k (t-1)}\, \langle \gamma - \bar\rho\,,\, f_k \rangle^2\;,$$

where $J = (1 - e^{-\lambda_1})^{-1}$.

Reporting the previous estimate to (6.17) yields that

$$I_{[0,1]}(w|\bar\rho) \;\le\; \Big(\frac{2J}{\delta_0}\Big)^2 \int_0^1 \sum_{k\ge 1} \lambda_k\, e^{2\lambda_k (t-1)}\, \langle \gamma - \bar\rho\,,\, f_k \rangle^2\, dt \\ \le\; \Big(\frac{2J}{\delta_0}\Big)^2 \sum_{k\ge 1} \langle \gamma - \bar\rho\,,\, f_k \rangle^2 \;=\; \Big(\frac{2J}{\delta_0}\Big)^2 \,\|\, \gamma - \bar\rho \,\|_2^2\;,$$

which concludes the proof of the lemma. □

We can now prove the upper bound for the quasi-potential and conclude the proof of Theorem 2.6.

Lemma 6.12. *For each $\gamma \in \mathcal{M}_{\rm ac}$, we have $V(\gamma) \le S(\gamma)$.*

Proof. Fix $0 < \varepsilon < (\delta_0/2)\min\{(1/4),(1/\Lambda)\}$, where Λ has been introduced in the statement of Lemma 6.11. Let $\gamma \in \mathcal{M}_{\rm ac}$, and recall that we denote by $v^{(\gamma)}(t,x)$ the solution to (6.8) with initial condition γ. By Lemma 6.10, there exists $T_1 = T_1(\varepsilon)$ such that $\|v^{(\gamma)}(t) - \bar\rho\|_\infty < \varepsilon$ for any $t \ge T_1$. Since $v^{(\gamma)}(T_1)$ fullfils the hypotheses of Lemma 6.11, let w be the path which connects $\bar\rho$ to $v^{(\gamma)}(T_1)$ in the interval $[0,1]$ constructed in that lemma.

Let $T := T_1 + 1$ and $w^*(t)$, $t \in [0,T]$, be the path

$$w^*(t) \;=\; \begin{cases} w(t) & \text{for } 0 \le t \le 1 \\ v^{(\gamma)}(T-t) & \text{for } 1 \le t \le T \end{cases} \tag{6.18}$$

Recall the definition of $\mathcal{M}_\delta$ given in (6.1). Let $(\gamma_n,\, n \ge 1)$ be a sequence such that $\gamma_n \in \mathcal{M}_{\delta_n}$ for some $\delta_n > 0$ and which converges to γ a. s. Denote by $v^{(\gamma_n)}$ the solution to (6.8) with initial condition γ_n.

Claim 1: $v^{(\gamma_n)}(T_1)$ converges to $v^{(\gamma)}(T_1)$ in $C([0,1])$.

To prove this claim, let $F = F(\gamma)$, $F_n = F(\gamma_n)$, and denote by $F^{(\gamma)}$, $F^{(\gamma_n)}$ the solutions to (1.1) with initial conditions F, F_n, respectively.

By Proposition 5.6, F_n converges to F in $C^1([0,1])$. Hence, by Lemma B.5, $F^{(\gamma_n)}(T_1)$ converges to $F^{(\gamma)}(T_1)$ in $C^2([0,1])$. On the other hand, by Lemma 6.6, there exists a constant $c_1 > 0$ such that $c_1 \le \nabla F^{(\gamma_n)}(T_1) \le c_1^{-1}$, $c_1 \le \nabla F^{(\gamma)}(T_1) \le c_1^{-1}$ for all $n \ge 1$. Hence, by (6.13), $v^{(\gamma_n)}(T_1)$ converges to $v^{(\gamma)}(T_1)$ in $C([0,1])$, as claimed.

Since $\|v^{(\gamma)}(T_1) - \bar\rho\|_\infty < \varepsilon$, by Claim 1, there exists n_0 such that $\|v^{(\gamma_n)}(T_1) - \bar\rho\|_\infty < 2\,\varepsilon$ for all $n \ge n_0$. Fix such n, and let $w^n(t)$ be the path joining $\bar\rho$ to $v^{(\gamma_n)}(T_1)$ in the time interval $[0,1]$ constructed in Lemma 6.11. Define the path $w^{n,*}(t)$, $0 \le t \le T_1 + 1$ as

$$w^{n,*}(t) \;=\; \begin{cases} w^n(t) & \text{for } 0 \le t \le 1 \\ v^{(\gamma_n)}(T-t) & \text{for } 1 \le t \le T \end{cases} \tag{6.19}$$

Claim 2: The path $w^{n,*}$ converges in $D([0,T],\mathcal{M})$ to w^*.

Before proving this claim, we conclude the proof of the lemma. By the lower semi-continuity of the functional $I_{[0,T]}(\,\cdot\,|\,\bar\rho)$,

$$I_{[0,T]}(w^*|\bar\rho) \;\le\; \liminf_n I_{[0,T]}(w^{n,*}|\bar\rho)\;. \tag{6.20}$$

On the other hand, by definition of the rate function and (4.1),

$$I_{[0,T]}\big(w^{n,*}\,|\,\bar\rho\big) \;\le\; I_{[0,1]}\big(w^n\,|\,\bar\rho\big) \;+\; I_{[0,T_1]}\big(v^{(\gamma_n)}(T_1-\cdot)\,|\,v^{(\gamma_n)}(T_1)\big)\;. \tag{6.21}$$

By Lemma 6.11, for $n \ge n_0$

$$I_{[0,1]}\big(w^n|\bar\rho\big) \;\le\; C_0\,\|v^{(\gamma_n)}(T_1) - \bar\rho\|_2^2 \tag{6.22}$$

for some constant $C_0 = C_0(\delta_0)$.

By Proposition 6.7 and Lemmata 6.8, 6.9, $(x,t) \mapsto v^{(\gamma_n)}(T_1 - t, x)$ belongs to $C^{1,2}([0,T_1]\times[0,1])$ and is bounded away from 0 and 1, namely it belongs to D_{T_1,δ_n} for some $\delta_n > 0$. Hence, by Lemma 6.5, as $v^{(\gamma_n)}(T_1 - \,\cdot\,)$ solves (6.6) with $K = 0$,

$$I_{[0,T_1]}\big(v^{(\gamma_n)}(T_1-\cdot)\,|\,v^{(\gamma_n)}(T_1)\big) \;=\; S_0(\gamma_n) \;-\; S_0(v^{(\gamma_n)}(T_1))\;. \tag{6.23}$$

By equations (6.20)–(6.23),

$$I_{[0,T]}(w^*\,|\,\bar\rho) \;\le\; \liminf_n \big\{\, S_0(\gamma_n) \;-\; S_0(v^{(\gamma_n)}(T_1)) \;+\; C_0\,\|v^{(\gamma_n)}(T_1) - \bar\rho\|_2^2 \,\big\}\;.$$

By Remark 5.8, $S_0(\gamma_n)$ converges to $S_0(\gamma)$. Thus, by the convergence of $v^{(\gamma_n)}(T_1)$ to $v^{(\gamma)}(T_1)$ in $C([0,1])$, the lower semicontinuity of S_0, and the bound $\|v^{(\gamma)}(T_1) - \bar\rho\|_\infty < \varepsilon$, the right-hand side is less than or equal to

$$S_0(\gamma) \;-\; S_0(\bar\rho) \;+\; C_0\, \|v^{(\gamma)}(T_1) - \bar\rho\|_2^2 \;\le\; S(\gamma) \;+\; C_0\, \varepsilon^2 \;.$$

To completes the proof of the lemma, it remains to show that Claim 2 is in force.

Proof of Claim 2. It is enough to show that $w^{n,*}$ converges to w^* in $C([0,T],\mathcal{M})$. Equivalently, to show that $v^{(\gamma_n)}$ converges to $v^{(\gamma)}$ in $C([0,T_1],\mathcal{M})$ and w^n converges to w in $C([0,1],\mathcal{M})$.

Fix $0<t_0<T_1$ small. By the arguments presented in the proof of Claim 1 and since the convergence in Lemma B.5 is uniform for $t\ge t_0$, $v^{(\gamma_n)}$ converges to $v^{(\gamma)}$ in $C([t_0,T_1]\times[0,1])$.

On the other hand, by Lemma 5.3, $\nabla F^{(\gamma_n)}(t)$ and $\nabla F^{(\gamma)}(t)$ are uniformly bounded, and, by Lemma 6.6, $\nabla v^{(\gamma_n)}(t)$ and $\nabla v^{(\gamma)}(t)$ are uniformly bounded in $[0,T_1]$. As $v^{(\gamma_n)}$, $v^{(\gamma)}$ are weak solutions to (6.11), for each $G\in C([0,1])$,

$$\lim_{t_0\downarrow 0}\, \limsup_{n}\, \sup_{t\in[0,t_0]} \big|\, \langle v^{(\gamma_n)}(t), G\rangle \;-\; \langle v^{(\gamma)}(t), G\rangle \,\big| \;=\; 0\;.$$

This concludes the proof that $v^{(\gamma_n)}$ converges to $v^{(\gamma)}$ in $C([0,T],\mathcal{M})$.

By (6.13) and Remark B.6, we can extend the result obtained in Claim 1 and show that $v^{(\gamma_n)}(T_1)$ converges to $v^{(\gamma)}(T_1)$ in $C^2([0,1])$. Hence, by the explicit formula (6.14), w^n converges to w in $C([0,1]\times[0,1])$. This completes the proof of Claim 2 and of the lemma. □

Appendix A. The Robin Laplacian

We present in this section some results on the Robin Laplacian needed in the previous section. Denote by Δ_R the Laplacian on $[0,1]$ with Robin boundary conditions, sometimes called the Robin Laplacian [25, Section 4.3].

Consider the eigenvalue problem

$$\begin{cases} -\,\Delta f = \lambda\, f\;, \\ (\nabla f)(0) \;=\; A^{-1}\, f(0)\;, \\ (\nabla f)(1) \;=\; -\,B^{-1} f(1)\;. \end{cases} \tag{A.1}$$

The equation $-\,\Delta f = \lambda\, f$ can be turned into a two-dimensional ODE which yields that the solutions to (A.1) are given by $f(x) = a\,[\,\cos(\sqrt{\lambda}x) + b\,\sin(\sqrt{\lambda}x)\,]$ for some a, $b\in\mathbb{R}$. The boundary conditions are satisfied if and only if

$$\tan\sqrt{\lambda} \;=\; (A+B)\,\frac{\sqrt{\lambda}}{\lambda AB - 1}\;, \tag{A.2}$$

in which case $b = (A\sqrt{\lambda})^{-1}$. This identity excludes $\lambda=0$ from the set of eigenvalues of the Robin Laplacian.

An analysis of (A.2) shows that it has a countable set of solutions $\{\lambda_j : j\ge 1\}$, where $0<\lambda_1$, $\lambda_j<\lambda_{j+1}$ and $\lambda_j\sim j^2$ in the sense that there exists $0<c_0<c_1<\infty$ such that

$$c_0\, j^2 \;\le\; \lambda_j \;\le\; c_1\, j^2\;. \tag{A.3}$$

Denote by $\{f_j : j \ge 1\}$ the associated orthonormal eigenvectors, which form a basis of $\mathcal{L}^2([0,1])$. By the previous analysis,

$$f_j(x) \;=\; a_j \,\Big\{\, \cos(\sqrt{\lambda_j} x) \;+\; \frac{1}{A\sqrt{\lambda_j}}\, \sin(\sqrt{\lambda_j} x) \,\Big\}\;, \tag{A.4}$$

where a_j is chosen for f_j to have $\mathcal{L}^2$-norm equal to 1. It can be shown that $|a_j| \le C_0$ for all $j \ge 1$, where C_0 is a finite constant depending only on A and B. Therefore, by (A.3),

$$\| f_j \|_\infty \;\le\; C_0\;, \quad \| \nabla^n f_j \|_\infty \;\le\; C_0\, (\lambda_j)^{n/2} \;\le\; C_0\, j^n \tag{A.5}$$

for all $j \ge 1$, $n \ge 1$.

A straightforward computation provides a formula for the Green function of the Robin Laplacian: Let $K_R : [0,1] \times [0,1] \to \mathbb{R}_+$ be given by

$$K_R(x,y) \;=\; \frac{1}{1+A+B} \begin{cases} (B+1-x)\,(A+y)\;, & 0 \le y \le x \le 1\;, \\ (B+1-y)\,(A+x)\;, & 0 \le x \le y \le 1\;. \end{cases} \tag{A.6}$$

Denote by K_R the integral operator defined by

$$(K_R f)(x) \;=\; \int_0^1 K_R(x,y)\, f(y)\; dy\;.$$

Then, $K_R = (-\Delta_R)^{-1}$.

Denote by $\mathcal{H}_R$ the Hilbert space obtained by completing the space $C^2_{A,B}([0,1]) = \{f \in C^2([0,1]) : (\nabla f)(0) \;=\; A^{-1}\, f(0)\;,\; (\nabla f)(1) \;=\; -\,B^{-1} f(1)\}$ endowed with the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}_R}$ defined by

$$\begin{aligned} \langle\, f\,,\, g\,\rangle_{\mathcal{H}_R} \;&=\; \langle\, f\,,\, (-\,\Delta_R) g\,\rangle \\ &=\; \frac{1}{A}\, f(0)\, g(0) \;+\; \int_0^1 (\nabla f)(x)\, (\nabla g)(x)\; dx \;+\; \frac{1}{B}\, f(1)\, g(1)\;. \end{aligned} \tag{A.7}$$

Denote by $\|f\|_{\mathcal{H}_R}$ the norm induced by the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}_R}$. We have that

$$\|f\|^2_{\mathcal{H}_R} \;=\; \sum_{k\ge 1} \lambda_k\, \langle f\,,\, f_k\rangle^2\;. \tag{A.8}$$

for all $f \in \mathcal{H}_R$.

The norms $\|\,\cdot\,\|_{\mathcal{H}_R}$ and $\|\,\cdot\,\|_{\mathcal{H}_1}$ are equivalent. There exist finite constants $0 < C_1 < C_2 < \infty$ such that

$$C_1\, \|f\|_{\mathcal{H}_1} \;\le\; \|f\|_{\mathcal{H}_R} \;\le\; C_2\, \|f\|_{\mathcal{H}_1} \tag{A.9}$$

for all $f \in C^2([0,1])$. In particular, the spaces $\mathcal{H}_R$ and $\mathcal{H}_1$ coincide.

In terms of the eigenfunctions f_k,

$$\|\, f\,\|^2_{\mathcal{H}_R} \;=\; \sum_{k\ge 1} \lambda_k\, |\, \langle\, f\,,\, f_k\,\rangle\, |^2\;. \tag{A.10}$$

Moreover, a straightforward computation yields that for all $f \in C^2_{A,B}([0,1])$,

$$\|\, f\,\|^2_\infty \;\le\; 2\,(A \vee 1)\, \|f\|^2_{\mathcal{H}_R}\;. \tag{A.11}$$

Fix a function f in $\mathcal{H}_1$. It is well known that there exists a continuous function $f^{(c)} : [0,1] \to \mathbb{R}$ (actually Hölder continuous, $|f^{(c)}(y) - f^{(c)}(x)| \le \|f\|_2 |y-x|^{1/2}$) such that $f = f^{(c)}$ almost surely. Moreover, for all $h \in C^1([0,1])$,

$$\int_0^1 f\, \nabla h\; dx \;=\; f^{(c)}(1)\, h(1) \;-\; f^{(c)}(0)\, h(0) \;-\; \int_0^1 \nabla f\, h\; dx\;. \tag{A.12}$$

The next result provides an explicit formula for $f^{(c)}$ in terms of the eigenvectors f_k.

Lemma A.1. *There exists a finite constant C_0 such that*

$$\sum_{k\geq 1} \big|\, \langle f\,,\, f_k\rangle \,\big| \;\le\; C_0\, \|\, f\,\|_{\mathcal{H}_R}$$

for all $f \in \mathcal{H}_1$. In particular, $\sum_{k\geq 1}\langle f\,,\, f_k\rangle\, f_k(\cdot)$ defines a continuous function, and, for almost all $x\in[0,1]$,

$$f(x) \;=\; \sum_{k\geq 1}\langle f\,,\, f_k\rangle\, f_k(x)\;. \tag{A.13}$$

Proof. By (A.9), f belongs to $\mathcal{H}_R$. By Schwarz inequality,

$$\Big(\sum_{k\geq 1} \big|\, \langle f\,,\, f_k\rangle\,\big|\Big)^2 \;\le\; \sum_{k\geq 1}\lambda_k \,\big|\, \langle f\,,\, f_k\rangle\,\big|^2 \sum_{k\geq 1}\frac{1}{\lambda_k}\;\cdot$$

The second sum is finite by (A.3) and the first one is finite by (A.10). This proves the first assertion.

Since each function f_k is continuous, and a summable sum of continuous functions is continuous, $\sum_{k\geq 1}\langle f\,,\, f_k\rangle\, f_k(\cdot)$ defines a continuous function. As $(f_k : k\geq 1)$ forms an orthonormal basis of $\mathcal{L}^2([0,1])$, $f = \sum_{k\geq 1}\langle f\,,\, f_k\rangle\, f_k$ as an identity in $\mathcal{L}^2([0,1])$. In particular, these functions are equal almost everywhere. □

Denote by $(P^{(R)}_t : t\geq 0)$ the semigroup in $\mathcal{L}^2([0,1])$ generated by the Robin Laplacian: For any function $f\in\mathcal{L}^2([0,1])$, $t>0$,

$$P^{(R)}_t f \;=\; \sum_{k\geq 1} e^{-\lambda_k t}\,\langle f\,,\, f_k\rangle\, f_k\;. \tag{A.14}$$

In particular, for each $t\geq 0$, $P^{(R)}_t$ is a symmetric operator in $\mathcal{L}^2([0,1])$ and $P^{(R)}_t f\in C^\infty([0,1])$ for all $f\in\mathcal{L}^2([0,1])$. Moreover, as $P^{(R)}_t$ is symmetric, by (A.10), $P^{(R)}_t$ is a contraction in $\mathcal{H}_R$ and $\mathcal{L}^2([0,1])$:

$$\begin{aligned} \|\, P^{(R)}_t f\,\|^2_{\mathcal{H}_R} \;&=\; \sum_{k\geq 1} e^{-2\lambda_k t}\,\lambda_k\,|\,\langle f\,,\, f_k\rangle\,|^2 \;\le\; \|\, f\,\|^2_{\mathcal{H}_R}\;, \\ \|\, P^{(R)}_t f\,\|^2_{2} \;&=\; \sum_{k\geq 1} e^{-2\lambda_k t}\,|\,\langle f\,,\, f_k\rangle\,|^2 \;\le\; \|\, f\,\|^2_{2}\;. \end{aligned} \tag{A.15}$$

Let $f\in\mathcal{L}^2([0,1])$ be given by $f=\sum_{k\geq 1}\langle f\,,\, f_k\rangle\, f_k$. For each $t>0$, there exists a finite constant $C_0(t)$ such that

$$\|\, P^{(R)}_t f\,\|^2_{\infty} \;\le\; C_0(t)\,\|\, f\,\|^2_2\;, \quad \|\, P^{(R)}_t f\,\|^2_{\mathcal{H}_R} \;\le\; C_0(t)\,\|\, f\,\|^2_2\;. \tag{A.16}$$

Indeed, by (A.10) and since $P^{(R)}_t$ is symmetric and $P^{(R)}_t f_k = e^{-\lambda_k t} f_k$,

$$\|\, P^{(R)}_t f\,\|^2_{\mathcal{H}_R} \;=\; \sum_{k\geq 1}\lambda_k\, e^{-2\lambda_k t}\big|\,\langle f\,,\, f_k\rangle\,\big|^2 \;\le\; C_0(t)\sum_{k\geq 1}\big|\,\langle f\,,\, f_k\rangle\,\big|^2 \;=\; C_0(t)\,\|\, f\,\|^2_2$$

for some finite constant $C_0(t)$. On the other hand, by Schwarz inequality and (A.5),

$$\|\, P^{(R)}_t f\,\|^2_{\infty} \;=\; \Big\|\sum_{k\geq 1} e^{-\lambda_k t}\,\langle f\,,\, f_k\rangle\, f_k\Big\|^2_\infty \;\le\; \sum_{k\geq 1} e^{-2\lambda_k t}\sum_{k\geq 1}\langle f\,,\, f_k\rangle^2 \;=\; C_0(t)\,\|\, f\,\|^2_2$$

for some finite constant $C_0(t)$.

Lemma A.2. *There exists a finite constant C_0 such that*

$$\| P_t^{(R)} f - f \|_2 \le C_0 \, t^{1/3} \, \| f \|_{\mathcal{H}_R}$$

for all $t \ge 0$, $f \in \mathcal{H}_R$.

Proof. Since $(f_k : k \ge 1)$ is an orthonormal basis of $\mathcal{L}^2([0,1])$,

$$\| P_t^{(R)} f - f \|_2^2 = \sum_{k \ge 1} \big[e^{-\lambda_k t} - 1 \big]^2 \, \big| \langle f , f_k \rangle \big|^2 .$$

Fix $k_0 \ge 1$. Since the sequence λ_k increases, the right-hand side can be bounded by

$$\big[e^{-\lambda_{k_0} t} - 1 \big]^2 \sum_{k=1}^{k_0 - 1} \big| \langle f , f_k \rangle \big|^2 + \frac{1}{\lambda_{k_0}} \sum_{k \ge k_0} \lambda_k \, \big| \langle f , f_k \rangle \big|^2 .$$

The first sum is bounded by $\| f \|_2^2$. In view of (A.10), the second one is bounded by $\| f \|_{\mathcal{H}_R}^2$ so that

$$\| P_t^{(R)} f - f \|_2^2 \le \big[1 - e^{-\lambda_{k_0} t} \big]^2 \, \| f \|_2^2 + \frac{1}{\lambda_{k_0}} \| f \|_{\mathcal{H}_R}^2 .$$

As $1 - e^{-x} \le x$, $x > 0$, and since, by (A.9), $\| f \|_2 \le C_0 \| f \|_{\mathcal{H}_r}$ for some finite constant C_0,

$$\| P_t^{(R)} f - f \|_2^2 \le \Big\{ C_0 \, (\lambda_{k_0} t)^2 + \frac{1}{\lambda_{k_0}} \Big\} \, \| f \|_{\mathcal{H}_R}^2 .$$

To complete the proof, it remains to choose k_0 such that $\lambda_{k_0}^{-3} \sim t^2$. □

Lemma A.3. *There exists a finite constant C_0 such that*

$$\| P_t^{(R)} f - f \|_\infty \le C_0 \, t^{1/5} \, \| f \|_{\mathcal{H}_R}$$

for all $t \ge 0$, $f \in C([0,1]) \cap \mathcal{H}_R$.

Proof. Fix $x \in [0,1]$. Since f is continuous, by (A.13) and (A.5),

$$\big\{ P_t^{(R)} f(x) - f(x) \big\}^2 \le C_0 \Big(\sum_{k \ge 1} \big[1 - e^{-\lambda_k t} \big] \, \big| \langle f , f_k \rangle \big| \Big)^2$$

for some finite constant C_0. By Schwarz inequality and (A.10), the right-hand is bounded by

$$C_0 \sum_{k \ge 1} \frac{1}{\lambda_k} \big[1 - e^{-\lambda_k t} \big]^2 \sum_{k \ge 1} \lambda_k \, \big| \langle f , f_k \rangle \big|^2 = C_0 \, \| f \|_{\mathcal{H}_R}^2 \sum_{k \ge 1} \frac{1}{\lambda_k} \big[1 - e^{-\lambda_k t} \big]^2 .$$

It remains to estimate the sum. Fix $k_0 \ge 1$. Since the sequence λ_k increases, as $1 - e^{-x} \le x$, $x > 0$, by (A.3), the sum is less than or equal to

$$C_0 \big[1 - e^{-\lambda_{k_0} t} \big]^2 + \sum_{k \ge k_0} \frac{1}{\lambda_k} \le C_0 \Big\{ (k_0^2 \, t)^2 + \frac{1}{k_0} \Big\}$$

for some finite constant C_0. It remains to choose k_0 such that $k_0^5 \sim t^{-2}$. □

Appendix B. Heat equations with mixed boundary conditions

We present in this section some result on the initial-boundary value problems (1.1), (4.3). Denote by $\mathcal{H}^1 = \mathcal{H}^1([0,1])$ the Hilbert space obtained by completing the space $C^1([0,1])$ endowed with the scalar product $\langle \cdot\,,\,\cdot \rangle_{\mathcal{H}^1}$ defined by

$$\langle f\,,\,g \rangle_{\mathcal{H}^1} \;=\; \langle f\,,\,g \rangle \;+\; \langle \nabla f\,,\,\nabla g \rangle\;. \tag{B.1}$$

Denote by $\|f\|_{\mathcal{H}^1}$ the norm induced by the scalar product $\langle \cdot\,,\,\cdot \rangle_{\mathcal{H}^1}$. Fix a function $\phi \in \mathcal{L}^2([0,1])$, and consider the initial-boundary problem

$$\begin{cases} \partial_t u \;=\; \Delta u \\ (\nabla u)(t,0) \;=\; A^{-1}\, u(t,0) \\ (\nabla u)(t,1) \;=\; -\, B^{-1}\, u(t,1) \\ u(0,\cdot) \;=\; \phi(\cdot)\;. \end{cases} \tag{B.2}$$

Definition B.1. *A function u in $\mathcal{L}^2([0,T];\mathcal{H}^1)$ is said to be a generalized (or weak) solution in the cylinder $[0,T]\times[0,1]$ of the equation* (B.2) *if*

$$\begin{aligned} &\int_0^1 u_t\, H_t\; dx \;-\; \int_0^1 \phi\, H_0\; dx \;-\; \int_0^t ds \int_0^1 u_s\, \partial_s H_s\; dx \\ &\quad =\; -\int_0^t ds \int_0^1 \nabla u_s\, \nabla H_s\; dx \;-\; \int_0^t \Big\{ \frac{1}{B}\, u_s(1)\, H_s(1) \;+\; \frac{1}{A}\, u_s(0)\, H_s(0) \Big\}\; ds \end{aligned}$$

for every $0 < t \le T$, function H in $C^{1,2}([0,T]\times[0,1])$.

Theorem B.2. *For each $\phi \in \mathcal{L}^2([0,1])$, there exists one and only one generalized solution to* (B.2)*. Moreover,*

- (a) *The solution is smooth in $(0,\infty)\times[0,1]$ and can be represented as $u(t,x) = (P^{(R)}_t \phi)(x)$, where $P^{(R)}_t$ is the semigroup associated to the Robin Laplacian.*
- (b) *For all $(t,x) \in \mathbb{R}_+ \times [0,1]$,*

$$\min\{\,0\,,\, \operatorname{ess\,inf} \phi\,\} \;\le\; u(t,x) \;\le\; \max\{\,0\,,\, \operatorname{ess\,sup} \phi\,\}\;. \tag{B.3}$$

- (c) *If $\phi(x) \le b$ for some $b>0$, then, for each $t_0>0$ there exists $\epsilon>0$ such that $u(t,x) \le b-\epsilon$ for all $(t,x)\in[t_0,\infty)\times[0,1]$. Analogously, if $\phi(x)\ge a$ for some $a<0$, then, for each $t_0>0$ there exists $\epsilon>0$ such that $u(t,x)\ge a+\epsilon$ for all $(t,x)\in[t_0,\infty)\times[0,1]$.*
- (d) *If ϕ belongs to $C([0,1])\cap\mathcal{H}_1$, then the solution belongs to $C([0,\infty)\times[0,1])$.*

Proof. Existence and uniqueness of generalized solutions, as well as their representation in terms of the semigroup $P^{(R)}_t$ is the content of Theorems 1 and 3 in [23, Section VI.2].

We turn to (B.3). Assume first that ϕ belongs to $\mathcal{H}_1$. By (A.9), $\phi \in \mathcal{H}_R$, and, by Lemma A.3, $u(t)$ converges to ϕ in $\mathcal{L}^\infty([0,1])$ as $t\to0$. Since the solution is smooth in $(0,\infty)\times[0,1]$, by the maximum principle stated in Theorems 2 and 3 of [24, Chapter 3],

$$\min\{\,0\,,\, \inf_{0\le y\le 1} u(t_0,y)\,\} \;\le\; u(t,x) \;\le\; \max\{\,0\,,\, \sup_{0\le y\le 1} u(t_0,y)\,\}$$

for all $(t,x)\in[t_0,\infty)\times[0,1]$. Letting $t_0\to0$, as $u(t_0)$ converges to ϕ in $\mathcal{L}^\infty([0,1])$, yields (B.3).

To extend this result to $\phi\in\mathcal{L}^2([0,1])$, we consider a sequence $\phi_n\in\mathcal{H}_1$ which converges to ϕ in $\mathcal{L}^2([0,1])$ and such that $\operatorname{ess\,inf}\phi \;\le\; \phi_n(x) \;\le\; \operatorname{ess\,sup}\phi$ for all

$0 \le x \le 1$. Denote by u^n the solution to (B.2) with initial condition ϕ_n. Fix $t > 0$. By the result for initial conditions in $\mathcal{H}_1$,

$$\begin{aligned} \min\{\, 0\,,\, \operatorname{ess\,inf} \phi \,\} \;&\le\; \min\{\, 0\,,\, \inf_{0\le y\le 1} \phi_n(y) \,\} \\ &\le\; u^n(t,x) \;\le\; \max\{\, 0\,,\, \sup_{0\le y\le 1} \phi_n(y) \,\} \;\le\; \max\{\, 0\,,\, \operatorname{ess\,sup} \phi \,\}\;. \end{aligned}$$

for all $0 \le x \le 1$. By (A.16), $u^n(t)$ converges to $u(t)$ in $\mathcal{L}^\infty([0,1])$. This completes the proof of (B.3).

Assume that $\phi(x) \le b$ for some $b > 0$. By (B.3), $u(t,x) \le b$ for all $t \ge 0$, $0 \le x \le 1$. Fix $t_0 > 0$, and assume that $\max_{0\le x\le 1} u(t_0,x) = b$. As $b > 0$, the boundary conditions imply that the maximum cannot be attained at the boundary. On the other hand, if it is attained at the interior, by Theorem 2 of [24, Chapter 3] and by the smoothness of the solution, $u(t,x) = b$ for all $(t,x) \in (0,t_0] \times [0,1]$. This is not possible at the boundary. Therefore, $\max_{0\le x\le 1} u(t_0,x) < b$. By the maximum principle, this bound can be extended to all $(t,x) \in [t_0,\infty) \times [0,1]$. The same argument applies to the lower bound.

Assertion (d) follows from Lemma A.3 and the representation of the solutions. □

It follows from the previous result that the operator $P^{(R)}_t$ is a contraction in $\mathcal{L}^\infty([0,1])$: for all $t \ge 0$, $f \in \mathcal{L}^\infty([0,1])$,

$$\| \, P^{(R)}_t f \, \|_\infty \;\le\; \| \, f \, \|_\infty \;. \tag{B.4}$$

Recall from (1.5) that we denote by $\bar\rho \in \mathcal{M}_{\rm ac}$ the unique stationary solution to the equation (1.1).

Definition B.3. *Fix $\gamma \in \mathcal{L}^2([0,1])$. A function u in $\mathcal{L}^2([0,T];\mathcal{H}^1)$ is said to be a generalized (or weak) solution in the cylinder $[0,T] \times [0,1]$ of the equation* (1.1) *if $u(t,x) - \bar\rho$ is a generalized solution to the initial-boundary problem* (B.2) *with initial condition $\gamma - \bar\rho$.*

Therefore, a function u in $\mathcal{L}^2([0,T];\mathcal{H}^1)$ is a generalized solution in the cylinder $[0,T] \times [0,1]$ of the equation (1.1) if

$$\begin{aligned} &\int_0^1 u_t\, H_t\, dx \;-\; \int_0^1 \gamma\, H_0\, dx \;-\; \int_0^t ds \int_0^1 u_s\, \partial_s H_s\, dx = \;-\int_0^t ds \int_0^1 \nabla u_s\, \nabla H_s\, dx \\ &\quad -\; \int_0^t \Big\{ \frac{1}{B}\, [\, u_s(1) - \beta\,]\, H_s(1) \;+\; \frac{1}{A}\, [\, u_s(0) - \alpha\,]\, H_s(0) \Big\}\, ds \end{aligned}$$

for every $0 < t \le T$, function H in $C^{1,2}([0,T] \times [0,1])$.

Theorem B.4. *Fix $\gamma \in \mathcal{L}^2([0,1])$. There exists a unique generalized solution to* (1.1)*. The solution is smooth in $(0,\infty) \times [0,1]$ and satisfies the bounds*

$$\min\{\, \alpha\,,\, \operatorname{ess\,inf} \gamma \,\} \;\le\; u(t,x) \;\le\; \max\{\, \beta\,,\, \operatorname{ess\,sup} \gamma \,\} \tag{B.5}$$

for all $(t,x) \in [0,\infty) \times [0,1]$. Moreover, if $\gamma \in \mathcal{M}_{\rm ac}$, for all $0 < t_0 \le T$ there exists $\epsilon > 0$ such that $\epsilon \le u(t,x) \le 1-\epsilon$ for all $(t,x) \in [t_0,\infty) \times [0,1]$. Finally, the solution is continuous in $[0,\infty) \times [0,1]$ if γ belongs to $C([0,1]) \cap \mathcal{H}_1$.

Proof. The proof of this result is similar to the one of Theorem B.2. □

Lemma B.5. *Let γ, $(\gamma_n : n \geq 1)$ be a sequence of density profiles in $\mathcal{M}_{\rm ac}$. Assume that γ_n converges to γ in $\mathcal{M}_{\rm ac}$. Let u, u^n be the weak solutions to* (1.1) *with initial conditions γ, γ_n, respectively. Then, for all $t_0 > 0$, $u^n(t)$ converges to $u(t)$ in $C^2([0,1])$ uniformly in $t \in [t_0, \infty)$.*

Proof. Fix $t_0 > 0$. By Definition B.3, Theorem B.2.(a) and (A.14), for every $t \geq 0$,

$$u^n(t) \,-\, u(t) \;=\; P^{(R)}_t(\gamma_n - \gamma) \;=\; \sum_{k\geq 1} e^{-\lambda_k t}\, \langle \gamma_n - \gamma \,,\, f_k\rangle\, f_k \;.$$

By this identity and the hypothesis, since γ, γ_n are bounded, $u^n(t)$ converges to $u(t)$ in $C([0,1])$, uniformly for $t \geq t_0$. Taking the Laplacian on both sides of this identity produces on the right-hand side an extra factor $-\lambda_k$. Hence, $\Delta u^n(t)$ converges to $\Delta u(t)$ in $C([0,1])$, uniformly for $t \geq t_0$. Uniform convergence in $C^2([0,1])$ follows from the two previous convergences. □

Remark B.6. *Fix $k \geq 1$. The same proof yields that, for all $t_0 > 0$, $u^n(t)$ converges to $u(t)$ in $C^{2k}([0,1])$, uniformly in $t \in [t_0, \infty)$.*

We conclude this section defining weak solutions to equation (4.3) and stating a result on existence and uniqueness.

Definition B.7. *Fix $\gamma \in \mathcal{L}^1([0,1])$, $H \in C^{0,1}([0,T]\times[0,1])$. A function u in $\mathcal{L}^2([0,T];\mathcal{H}^1)$ is said to be a generalized, or weak, solution in the cylinder $[0,T]\times[0,1]$ of the equation* (4.3) *if*

$$\begin{aligned}
&\int_0^1 u_t\, G_t\; dx \;-\; \int_0^1 \gamma\, G_0\; dx \;-\; \int_0^t ds \int_0^1 u_s\, \partial_s G_s\; dx \\
&\quad =\; \int_0^t ds \int_0^1 \big\{ -\, \nabla u_s\, \nabla G_s \;+\; 2\,\sigma(u_s)\, \nabla H_s\, \nabla G_s \big\}\; dx \\
&\quad +\; \int_0^t \big\{ \mathfrak{p}_{\beta,B}(u_s(1), H_s(1))\, G_s(1) \;+\; \mathfrak{p}_{\alpha,A}(u_s(0), H_s(0))\, G_s(0) \big\}\; ds
\end{aligned} \tag{B.6}$$

for every $0 < t \leq T$ and function G in $C^{1,2}([0,T]\times[0,1])$.

Next result is taken from [19]

Theorem B.8. *Fix $\gamma \in \mathcal{M}_{\rm ac}$ and H in $C^{0,1}([0,T]\times[0,1])$. Then, there exists a unique weak solution to* (4.3). *Moreover, $0 \leq u(t,x) \leq 1$ a.s. in $[0,T]\times[0,1]$.*

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