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Hydrodynamic Limit of Boundary Driven Exclusion Processes with Nonreversible Boundary Dynamics

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Abstract
Using duality techniques, we derive the hydrodynamic limit for one-dimensional, boundary-driven, symmetric exclusion processes with different types of non-reversible dynamics at the boundary, for which the classical entropy method fails.

1 Introduction

Boundary-driven exclusion models can provide good examples of simple, solvable non-equilibrium models (see [1] and references therein). Such processes can exhibit rich behavior, depending on the nature on the boundary dynamics selected. One key goal in the study of these models is the derivation for nonequilibrium models of the large deviation functionals which plays the role the entropy does for equilibrium models (cf. [2–4]). Although progress has been made for specific models [5], as a first step to achieve this program for general classes of models, it is necessary to study both the stationary state and the hydrodynamic behavior of such models.

In [6], we investigated the stationary state for three classes of one-dimensional dynamics, whose bulk dynamics is symmetric simple exclusion (SSEP), and driven out of equilibrium by non-reversible, non-conservative dynamics at the boundaries. Classical tools can in some cases be adapted to derive the hydrodynamic limit for boundary-driven one-dimensional models for which the boundary dynamics is either reversible w.r.t. a product measure, or sped up or slowed down w.r.t. the bulk dynamics (cf. [7]). However, to the best of our knowledge, no hydrodynamic limit has been derived for models whose boundary evolves on the same time scale as the bulk and whose dynamics is not reversible with respect to a product measure.

In this article, we expand on the results obtained in [6], and use duality techniques to derive the hydrodynamic limit for two of the three classes of models investigated in [6]. Duality

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properties have been extensively used to derive hydrodynamic behaviors, in particular for SSEP dynamics (see [8] and references therein). One central challenge in using duality to derive hydrodynamic limits lies in closing the discrete difference equations satisfied by the n-point correlation functions (e.g. [9] for the Glauber + Kawasaki dynamics). In the first class of dynamics studied in this article, the boundary Markov generator preserves polynomials of degree one and two in the configuration, and therefore the equations for the density and correlation fields are naturally closed. In the second class, particles are created and annihilated at the left boundary at a rate which depends in a weak way on the local configuration at the left boundary (cf. (A2)). This ensures that the dual branching process ultimately dies, and that we are therefore indeed able to close the equations (cf. [6]). In both cases, the bulk dynamics is symmetric simple exclusion, and the right boundary is in contact with a reservoir at density $\beta \in (0, 1)$.

Note that, although the method used in this article applies to the third class of models investigated in [6], in which the left boundary dynamics is sped up by an extra factor $\ell_N \to \infty$, to derive the hydrodynamic limit they also require ℓ_N to be at least of order N. In this case, however, an adaptation of the more classical *entropy method* [10] can also be used, so that we do not consider this third class here. Because this article is based on duality argument, one main drawback of our method is that it is really mainly adapted to models with stirring dynamics in the bulk. Furthermore, the question of what happens when the creation/annihilation rate at the left boundary strongly depends on the configuration (i.e. when Assumption (A2) fails) remains open. It is to be noted that more recently, a general theory has been developed for the use of duality [11] in the context of interacting particle systems, applied in particular to so called inclusion dynamics [12] and asymmetric exclusion models to derive hydrodynamic limits. In this article, however, we do not need such elaborate tools, mainly because our bulk dynamics, namely simple exclusion, yields fairly simple equations for the density and correlation fields.

This article is organized as follows; in Sect. 2, we introduce the model, as well as the class of left boundary conditions to which our result applies. In Sect. 3, we carefully estimate the density at the left boundary, as well as the evolution of the density's gradient at each boundary. In Sect. 4, we estimate the correlation function of the dynamics. We conclude in Sect. 5 the proof of our main result using the estimates obtained in Sects. 3 and 4.

2 Notations and Main Results

2.1 General Notations

Consider $\Lambda_N = \{1, \ldots, N-1\}$, and let $\Omega_N = \{0, 1\}^{\Lambda_N}$. our the set of configurations on Λ_N. Elements of Ω_N will be denoted η, and for $j \in \Lambda_N$, $\eta_j = 1$ (resp. 0) is to be understood as site j being occupied (resp. empty) in η. We study in this article a Markov chain on Ω_N whose generator can be written

$$L_N = N^2(L_{r,N} + L_{b,N} + L_{l,N}). \tag{2.1}$$

The generator $L_{b,N}$ encompasses the bulk dynamics, symmetric simple exclusion, on Λ_N. More precisely, for any function $f : \Omega_N \to \mathbb{R}$, and any configuration $\eta \in \Omega_N$,

$$(L_{b,N}f)(\eta) = \sum_{k=1}^{N-2} \left\{ f(\sigma^{k,k+1}\eta) - f(\eta) \right\},$$

where $\sigma^{k,l}\eta$ is the configuration obtained from η by swapping the occupation variables η_k, η_l,

$$(\sigma^{k,l}\eta)_j = \begin{cases} \eta_l & \text{if } j = k \\ \eta_k & \text{if } j = l \\ \eta_j & \text{if } j \in \Lambda_N \setminus \{k,l\} \end{cases}.$$

At both boundaries, the dynamics is put in contact with non-conservative dynamics. On the right, the dynamics is coupled to a reservoir at density $\beta \in (0,1)$

$$(L_{r,N}f)(\eta) = [\beta(1 - \eta_{N-1}) + (1 - \beta)\eta_{N-1}]\left\{f\left(\sigma^{N-1}\eta\right) - f(\eta)\right\},$$

where for $k \in \Lambda_N$, $\sigma^k\eta$ is the configuration where the state of site k has been flipped,

$$(\sigma^k\eta)_j = \begin{cases} 1 - \eta_k & \text{if } j = k \\ \eta_j & \text{if } j \in \Lambda_N \setminus \{k\} \end{cases}.$$

Note that we choose at the right boundary a very simple dynamics (coupling with a large reservoir at equilibrium). However, our method still applies if the right boundary generator is chosen according to either of the two classes of dynamics introduced below.

Fix $p \in \mathbb{N}$, we denote $\Lambda_p^* = \{1, \ldots, p\}$ the microscopic set that plays the role of left boundary for Λ_N. The left boundary generator is written $L_{l,N} = L_R + L_C + L_A$, where

$$(L_R f)(\eta) = \sum_{j \in \Lambda_p^*} r_j \left[\alpha_j(1 - \eta_j) + \eta_j(1 - \alpha_j)\right]\left\{f\left(\sigma^j\eta\right) - f(\eta)\right\}, \tag{2.2}$$

$$(L_C f)(\eta) = \sum_{j \neq k \in \Lambda_p^*} c_{j,k} \left[\eta_j(1 - \eta_k) + \eta_k(1 - \eta_j)\right]\left\{f\left(\sigma^j\eta\right) - f(\eta)\right\}, \tag{2.3}$$

$$(L_A f)(\eta) = \sum_{j \neq k \in \Lambda_p^*} a_{j,k} \left[\eta_j\eta_k + (1 - \eta_j)(1 - \eta_k)\right]\left\{f\left(\sigma^j\eta\right) - f(\eta)\right\}, \tag{2.4}$$

and $(r_j)_{j \in \Lambda_p^*}$, $(c_{j,k})_{j \neq k \in \Lambda_p^*}$, $(a_{j,k})_{j \neq k \in \Lambda_p^*}$ are non-negative constants. The $(\alpha_j)_{j \in \Lambda_p^*}$ are in $[0,1]$, and are the respective densities of each of the reservoirs linked to sites $1 \leq j \leq p$. The $c_{j,k}$'s (resp. $a_{j,k}$'s) are to be understood as copy (resp. anticopy) rates, at which site j takes the value (resp. the inverse of the value) of site k. The r_j's are reservoir rates, at which site j is updated according to a reservoir at density α_j. Note that the stirring generator (2.1) occurs in Λ_N, therefore it also affects the left boundary Λ_p^*, as well as links Λ_p^* with $\Lambda_N \setminus \Lambda_p^* = \{p+1, \ldots, N-1\}$. We prove in Lemma 3.3 of [6] that, assuming

$$\sum_{j \neq k \in \Lambda_p^*} a_{j,k} + \sum_{j \in \Lambda_p^*} r_j > 0, \tag{A1}$$

the generator $L_{l,N} + L_{b,p+1}$ (where $(L_{b,p+1}f)(\eta) = \sum_{k=1}^{p-1}\{f(\sigma^{k,k+1}\eta) - f(\eta)\}$ is the stirring generator limited to jumps in Λ_p^*) admits a unique invariant measure μ (which does not depend on N). We denote

$$\alpha := \mathbb{E}_\mu(\eta_p). \tag{2.5}$$

As investigated in [6], the non-conservative dynamics encoded in $L_{l,N}$ macroscopically behaves as a reservoir at density α. We will not consider the case $\sum_{j \neq k \in \Lambda_p^*} a_{j,k} + \sum_{j \in \Lambda_p^*} r_j =$

0, $\sum_{j \neq k \in \Lambda_p^*} c_{j,k} > 0$, in which $L_{l,N} + L_{b,p}$ admits two degenerate stationary states respectively concentrated on the full and empty configurations.

In this article, we will focus on the case

$$\sum_{j \neq k \in \Lambda_p^*} a_{j,k} = 0, \quad \text{and} \quad \sum_{j \in \Lambda_p^*} r_j > 0.$$

This is purely for convenience: the case $\sum_{j \neq k \in \Lambda_p^*} a_{j,k} > 0$ offers no further difficulty w.r.t. the hydrostatic limit, so that the hydrodynamic limit in this case can be quite easily recovered from the present article and the tools introduced in [6].

Fix a smooth initial density profile $\rho_0 \in C^2([0, 1])$, and denote by ν_N the product measure on Ω_N close to the profile ρ_0

$$\nu_N(\eta) = \prod_{k \in \Lambda_N} [\eta_k \rho_0(k/N) + (1 - \eta_k)(1 - \rho_0(k/N))].$$

Let $D(\mathbb{R}_+, \Omega_N)$ the space of right-continuous functions $\eta : \mathbb{R}_+ \to \Omega_N$ with left limits. Denote by $\mathbb{P}_{\nu_N}$ the distribution on $D(\mathbb{R}_+, \Omega_N)$ induced by the process $\eta(t)$ started from ν_N, and driven by the generator L_N. Expectation with respect to $\mathbb{P}_{\nu_N}$ is denoted $\mathbb{E}_{\mu_N}$. We are now ready to state our main result.

Theorem 2.1 *Fix $T > 0$, and assume that (A1) holds. For any continuous function $G : [0, 1] \to \mathbb{R}$, and any $t \in [0, T]$*

$$\lim_{N \to \infty} \mathbb{E}_{\nu_N} \left(\left| \frac{1}{N} \sum_{k \in \Lambda_N} G(k/N) \, \eta_k(t) - \int_{[0,1]} G(u) \bar{\rho}(t, u) \, du \right| \right) = 0 \,,$$

where $\bar{\rho}$ is the unique solution of the linear elliptic equation

$$\begin{cases} \partial_t \rho(t, u) = \Delta \rho(t, u) & \text{for any } (t, u) \in]0, T] \times]0, 1[\\ \rho(0, \cdot) = \rho_0(\cdot) & \\ \rho(t, 0) = \alpha, \quad \rho(t, 1) = \beta & \text{for any } t \in]0, T] \end{cases} \quad (2.6)$$

where α was defined in (2.5).

Remark 2.2 As shown in [6], this choice for the left boundary generator $L_{l,N}$ is the most general for which we can write

$$L_{l,N} \eta_j = q^{1,j} + \sum_{k \in \Lambda_p^*} q_k^{1,j} \eta_k$$

and

$$L_{l,N} \eta_j \eta_k = q^{2,j,k} + \sum_{l \in \Lambda_p^*} q_l^{2,j,k} \eta_l + \sum_{l \neq m \in \Lambda_p^*} q_{l,m}^{2,j,k} \eta_l \eta_m,$$

for some constants $q^{1,j}$, $q_k^{1,j}$, $q^{1,j}$, $q^{2,j,k}$ $q_l^{2,j,k}$ and $q_{l,m}^{2,j,k}$. In other words, this model is the most general for which $L_{l,N}$ preserves polynomials of degree ν in η.

When this condition is not respected, one can still derive a hydrodynamic limit, if at the left boundary, particles are created and removed at a rate which depends in a small measure on the configuration at the boundary. This is the content of the next section.

2.2 Creation/Annihilation Rate Depending on the Local Boundary Configuration

In order to present as general a result as possible, we now change the left boundary generator $L_{l,N}$, to one where particles are created and annihilated at the first site depending on the state of the boundary. We therefore let

$$(\widetilde{L}_{l,N} f)(\eta) = c(\eta_1, \dots, \eta_p) \left\{ f(\sigma^1 \eta) - f(\eta) \right\} . \tag{2.7}$$

where c is a function $c : \{0, 1\}^p \to \mathbb{R}_+$.

The dynamics for this model is more general, however in order to derive the hydrodynamic limit, we need to assume that the creation and annihilation rate c do not depend too much on the boundary configuration. Let us denote by ξ the elements of $\{0, 1\}^{p-1}$, we let

$$A = \inf_{\xi \in \{0,1\}^{p-1}} c(0, \xi)$$

$$B = \inf_{\xi \in \{0,1\}^{p-1}} c(1, \xi)$$

the minimal creation and annihilation rates. Denote $\lambda(0, \xi) = c(0, \xi) - A$ and $\lambda(1, \xi) = c(1, \xi) - B$, we assume that

$$(p - 1) \sum_{\xi \in \{0,1\}^{p-1}} \{\lambda(0, \xi) + \lambda(1, \xi)\} \le A + B. \tag{A2}$$

We now state the hydrodynamic limit for this second model. We use analogous notations as for Theorem 2.1, and denote $\widetilde{\mathbb{P}}_{\nu_N}$ the distribution on $D(\mathbb{R}_+, \Omega_N)$ induced by the process $\eta(t)$ started from ν_N, and driven by the generator $\widetilde{L}_N := \widetilde{L}_{l,N} + L_{b,N} + L_{r,N}$. Expectation with respect to $\widetilde{\mathbb{P}}_{\nu_N}$ is denoted $\widetilde{\mathbb{E}}_{\mu^N}$.

Theorem 2.3 *Assume* (A2), *there exists $\widetilde{\alpha} \in [0, 1]$ such that for any $T > 0$, any continuous function $G : [0, 1] \to \mathbb{R}$, and any $t \in [0, T]$*

$$\lim_{N \to \infty} \widetilde{\mathbb{E}}_{\nu_N} \left(\left| \frac{1}{N} \sum_{k \in \Lambda_N} G(k/N) \, \eta_k(t) - \int_{[0,1]} G(u) \bar{\rho}(t, u) \, du \right| \right) = 0 ,$$

where $\bar{\rho}$ is the unique solution of (2.6), *except with $\widetilde{\alpha}$ replacing α.*

Note that the left density $\widetilde{\alpha}$ is the limit α introduced in Theorem 2.4 of [6]. We will only write the proof of Theorem 2.1 and assume that $L_A = 0$. With the tools developed for the hydrostatic limit in [6], the proof of Theorem 2.1 extends straightforwardly to both the case $\sum_{j \ne k \in \Lambda_p^*} a_{j,k} = 0$ and Theorem 2.3.

2.3 Duality and Scheme of the Proof

Denote for any $(t, k, l) \in [0, T] \times \Lambda_N^2$

$$\rho_N(t, k) = \mathbb{E}_{\nu_N}(\eta_k(t)) \tag{2.8}$$

the density at site $k \in \lambda_N$, and adopt a similar notation for the two-points correlation function

$$\varphi_N(t, k, l) = \mathbb{E}_{\nu_N} \left(\{\eta_k(t) - \rho_N(t, k)\}\{\eta_l(t) - \rho_N(t, l)\} \right). \tag{2.9}$$

To prove Theorem 2.1, we will use duality between ρ_N, (resp. φ_N), and random walks on Λ_N (resp. Λ_N^2).

We start by introducing a set of cemetery states

$$\partial \Lambda_N = \{\mathfrak{d}_1, \ldots, \mathfrak{d}_p\} \cup \{N\},$$

each representing one of the reservoirs, and let $\bar{\Lambda}_N = \Lambda_N \cup \partial \Lambda_N$. Further define the function $\rho_\mathfrak{d}$ on $\partial \Lambda_N$ given by

$$\rho_\mathfrak{d}(\mathfrak{d}_j) = \alpha_j, \ \forall j \in \Lambda_p^* \quad \text{and} \quad \rho_\mathfrak{d}(N) = \beta.$$

We extend the function ρ_N defined in (2.8) to $[0, T] \times \bar{\Lambda}_N$ by letting for any $t \geq 0$

$$\rho_N(t, \cdot) = \rho_\mathfrak{d}(\cdot) \quad \text{on} \ \partial \Lambda_N. \tag{2.10}$$

We now introduce dual generators, acting on functions on $\bar{\Lambda}_N$, defined by

$$(L_{b,N}^\dagger f)(j) = \begin{cases} (\Delta_N f)(j) := f(j+1) + f(j-1) - 2f(j) & \text{for } 1 < j < N, \\ (\nabla_N^+ f)(j) := f(j+1) - f(j) & \text{for } j = 1, \\ (\nabla_N^- f)(j) := f(j-1) - f(j) & \text{for } j = N-1, \end{cases} \tag{2.11}$$

$$(L_{r,N}^\dagger f)(j) = \mathbf{1}_{\{j=N-1\}} \{f(N) - f(j)\}$$

$$(L_R^\dagger f)(j) = \mathbf{1}_{\{j \in \Lambda_p^*\}} r_j \{f(\mathfrak{d}_j) - f(j)\} \tag{2.12}$$

and finally

$$(L_C^\dagger f)(j) = \mathbf{1}_{\{j \in \Lambda_p^*\}} \sum_{k \neq j \in \Lambda_p^*} c_{j,k} \{f(k) - f(j)\}. \tag{2.13}$$

Note in particular that any of the cemetery states in $\partial \Lambda_N$ is an absorbing state for each of these dual generators. Then, letting

$$L_N^\dagger = L_R^\dagger + L_C^\dagger + L_{b,N}^\dagger + L_{r,N}^\dagger, \tag{2.14}$$

using the fact that $\partial_t \rho_N(t, k) = N^2 E_{\nu_N}(L_N \eta_k(t))$ and notation (2.10), one obtains after elementary computations that the function ρ_N defined in (2.8) is a solution of the system

$$\begin{cases} \partial_t f = N^2 L_N^\dagger f \\ f(0, \cdot) = \rho_0(\cdot/N) & \text{on } \Lambda_N \\ f(t, \cdot) = \rho_\mathfrak{d}(\cdot), & \text{on } \partial \Lambda_N, \ \forall t \in [0, T] \end{cases}. \tag{2.15}$$

The first ingredient to prove Theorem 2.1 is showing that for any $t = t(N)$ large enough,

$$\rho_N(t, p+1) = \alpha + o_N(1), \tag{2.16}$$

where α is given by (2.5). Since ρ_N is solution of (2.15), and since on $\{p+1, \ldots, N-1\}$ $L_N^\dagger$ acts as the discrete Laplacian Δ_N (with our notation for site N, $(\{L_{r,N}^\dagger + L_{b,N}^\dagger\} f)(N-1) = (\Delta_N f)(N-1)$), this yields

$$\begin{cases} \partial_t \rho_N(t, k) = N^2 (\Delta_N \rho_N)(t, k) & \forall (t, k) \in [0, T] \times \{p+2, N-1\} \\ \rho_N(0, \cdot) = \rho_0(\cdot/N) & \text{on } \{p+2, \ldots, N-1\} \\ \rho_N(t, p+1) = \alpha + o_N(1) & \forall t \in]0, T] \\ \rho_N(t, N) = \beta & \forall t \in]0, T], \end{cases} \tag{2.17}$$

whose solution converges weakly as $N \to \infty$ towards the solution of (2.6). Proving (2.16) is the purpose of Sect. 3.

The second ingredient is a control of the two-points correlation function φ_N defined in (2.9): for some large set $S_{N,\delta} \subset \{(k,l), \quad p+1 \leq k < l \leq N-1\}$

$$\limsup_{\substack{N\to\infty \\ (k,l)\in S_{N,\delta} \\ t\in[0,T]}} \sup |\varphi_N(t,k,l)| = 0,$$

which allows, in Theorem 2.1, to replace $\eta_k(t)$ by its expectation $\rho_N(t,k)$. This estimate is obtained in Sect. 4. With these two elements, a few technical difficulties remain to prove Theorem 2.1, which is done in Sect. 5.

3 Estimation of the Left Density and Technical Lemmas

3.1 Estimation of the Density at the Boundaries

Define a continuous time random walk $\widetilde{X}$ on $\bar{\Lambda}_N$ driven by the sped-up dual generator $N^2 L_N^\dagger$ defined in (2.14). For any set $B \subset \bar{\Lambda}_N$, define $\widetilde{H}(B)$ as $\widetilde{X}$'s hitting time of the set B,

$$\widetilde{H}(B) = \inf\{s \geq 0, \widetilde{X} \in B\},$$

and define $\widetilde{H}_t(B) = t \wedge \widetilde{H}(B)$. For any $j \in \bar{\Lambda}_N$, denote $\widetilde{\mathbb{P}}_j$ and $\widetilde{\mathbb{E}}_j$ the distribution of $\widetilde{X}$ started from j and its expectation. Since ρ_N is solution of (2.15), it is well known that for any $j \in \bar{\Lambda}_N$ and any $t \geq 0$

$$\rho_N(t,j) = \widetilde{\mathbb{E}}_j \left[b_N\big(t - \widetilde{H}_t, \widetilde{X}(\widetilde{H}_t)\big)\right],$$

where we shortened $\widetilde{H}_t := \widetilde{H}_t(\partial\Lambda_N)$ and where b_N is the function giving the value of ρ_N at the space-time boundary, defined as

$$b_N(t,j) = \rho_0(j/N)\mathbf{1}_{\{t=0, j\in\Lambda_N\}} + \rho_{\mathfrak{d}}(j)\mathbf{1}_{\{j\in\partial\Lambda_N\}}. \tag{3.1}$$

To present the proof in as simple a setting as possible, however, it is not convenient that $L_N^\dagger$ has absorbing states. We therefore define $\bar{L}_N^\dagger = L_N^\dagger + L_{\mathfrak{d},N}^\dagger$, with

$$(L_{\mathfrak{d},N}^\dagger f)(j) = \mathbf{1}_{\{j\in\partial\Lambda_N\}}\left\{f(p+1) - f(j)\right\},$$

which allows jumps at rate 1 from any of the cemetery states to site $p+1$. We denote X a random walk driven by $N^2 \bar{L}_N^\dagger$, in particular, assuming that both random walk start from the same point in Λ_N, X coincides with $\widetilde{X}$ at least up until time

$$H(\partial\Lambda_N) = \inf\{s \geq 0, \ X \in \partial\Lambda_N\} = \widetilde{H}(\partial\Lambda_N).$$

We denote without " $\sim$ " all quantities and items relative to X, and thanks to the last observation, we are still able to write

$$\rho_N(t,j) = \mathbb{E}_j \left[b_N\big(t - H_t, X(H_t)\big)\right], \tag{3.2}$$

where once again $H_t := H_t(\partial\Lambda_N)$

We are now ready to state the main result of this section. We start by proving equation (2.16), in order to obtain the differential system (2.17). Since we will also need to control the density gradient at the boundaries $p+1$ and $N-1$ to estimate the correlations, we prove a more general result than (2.16), and estimate carefully the left and right densities.

Proposition 3.1 *For any $\varepsilon > 0$, there exists a constant $K = K(\varepsilon)$ such that for any time $s \in]0, T]$,*

$$\sup_{t \in [s,T]} |\rho_N(t, p+1) - \alpha| \le K \frac{N^{\varepsilon - 1}}{s}, \tag{3.3}$$

$$\sup_{t \in [s,T]} |\rho_N(t, p+1) - \rho_N(t, p+2)| \le K \frac{N^{\varepsilon - 1}}{s}, \tag{3.4}$$

and

$$\sup_{t \in [s,T]} |\beta - \rho_N(t, N-1)| \le K \frac{N^{\varepsilon - 1}}{s}. \tag{3.5}$$

Proof of Proposition 3.1 We will only detail the proof for the first identity, since the second is and third are proved in the same way. To estimate $\rho_N(t, p+1)$, we use (3.2), and start the random walk X at site $p+1$. Then, it performs excursions away from $p+1$, either in the bulk, in which case the excursion lasts a macroscopic time of order $1/N$ (recall that the whole random walk is accelerated by N^2), or in the left boundary, in which case it has a positive probability π to reach one of the cemetery states ∂_j's. In a time s, X will perform a number of excursions at least of order sN, each yielding a chance of ending in one of the cemetery states.

We now make this argument rigorous. Since similar proofs will be used repeatedly, we detail the proof of this Proposition, and will be more concise later on. Recall that X can jump from $\partial \Lambda_N$ to $p+1$ at rate 1, let $t_0 = 0 < t_1 < t_2 \cdots$ denote the successive arrival times of X at the site $p+1$:

$$t_0 = 0, \quad \text{and} \quad t_{n+1} = \inf\{t > t_n, \quad X(t) = p+1 \quad \text{and} \quad X(t^-) \ne p+1\}.$$

The random walk X being a Markov process, under $\mathbb{P}_{p+1}$, the successive excursions $(X(t))_{t_n \le t < t_{n+1}}$ are i.i.d. in $n \ge 0$. To distinguish the two types of excursions away from $p+1$, denote

$$E_n = \{X(t_{n+1}^-) = p+2\}$$

which indicates the nth excursion was performed in the bulk rather than in the boundary. Let us denote

$$F_n = \{ \text{There exists } (t, j) \in [t_n, t_{n+1}[\times \Lambda_p^*, \text{ such that } X(t) = \partial_j\}.$$
$$(\text{resp. } G_n = \{ \text{There exists } t \in [t_n, t_{n+1}[, \text{ such that } X(t) = N\}),$$

which indicates the n-th excursion reached one of the cemetery states ∂_n (resp. N). Finally, we denote by $d_n = t_{n+1} - t_n$ the duration of the n-th excursion. Since the excursions away from $p+1$ are i.i.d. under $\mathbb{P}_{p+1}$, so are the $(E_n)_{n \in \mathbb{N}}$, $(F_n)_{n \in \mathbb{N}}$, $(G_n)_{n \in \mathbb{N}}$, and $(d_n)_{n \in \mathbb{N}}$.

Define

$$\pi = \mathbb{P}_p\Big[H(\partial \Lambda_N) < H(\{p+1\}) \Big], \tag{3.6}$$

which is the probability that an excursion in the left boundary reaches one of the cemetery states before coming back to site $p+1$. One easily obtains, for any $n \ge 0$, the following properties:

$$\mathbb{P}_{p+1}(E_n) = \mathbb{P}_{p+1}(E_n^c) = 1/2, \tag{3.7}$$

$$\mathbb{P}_{p+1}(F_n \mid E_n) = 0 \quad \text{and} \quad \mathbb{P}_{p+1}(F_n \mid E_n^c) = \pi, \tag{3.8}$$

$$\mathbb{P}_{p+1}(G_n \mid E_n) = \frac{1}{N-1-p} \quad \text{and} \quad \mathbb{P}_{p+1}(G_n \mid E_n^c) = 0. \tag{3.9}$$

Furthermore, there exists a constant C such that

$$\mathbb{E}_{p+1}(d_n \mid E_n) \leq C/N \quad \text{and} \quad \mathbb{E}_{p+1}(d_n \mid E_n^c) \leq C/N^2. \tag{3.10}$$

Because of the second part of this equation, the constant C depends a priori on the rates of the left boundary dual generator $L_{l,N}^\dagger$. These identities are elementary, we do not detail their proof. In particular, (3.10) uses the fact that the generator $L_N^\dagger$ was accelerated by N^2, therefore a excursion in the bulk as a duration of order $1/N$, whereas in the boundary, the random walk will perform a finite number of steps before heading back to site $p+1$, so that the time length of a typical excursion is of order N^{-2}.

We now prove (3.3). Fix $t \in [0, T]$, (3.2) yields

$$\begin{aligned}
\rho_N(t, p+1) &= \mathbb{E}_{p+1}\left[b_N\left(t - H_t(\partial\Lambda_N), X(H_t(\partial\Lambda_N))\right)\right] \\
&= \mathbb{E}_{p+1}\left[\rho_0(X(t/n))\mathbf{1}_{\{H_t(\partial\Lambda_N)=t\}}\right] + \beta\mathbb{P}_{p+1}\left[H_t(\partial\Lambda_N) = H(\{N\})\right] \\
&\quad + \sum_{j\in\Lambda_p^*} \alpha_j\mathbb{P}_{p+1}\left[H_t(\partial\Lambda_N) = H(\{\mathfrak{d}_j\})\right],
\end{aligned}$$

so that, since ρ_0, β and the α_j's are less than 1,

$$\left|\rho_N(t, p+1) - \sum_{j\in\Lambda_p^*} \alpha_j\mathbb{P}_{p+1}\left[H(\partial\Lambda_N) = H(\{\mathfrak{d}_j\})\Big| H(\partial\Lambda_N \setminus \{N\}) \leq H(\{N\})\right]\right|$$

$$\leq C\mathbb{P}_{p+1}\left[H_t(\partial\Lambda_N) = H_t(\{N\})\right] \tag{3.11}$$

for some constant $C := C(p)$.

Let us now estimate the right-hand side above. From Eqs. (3.7), (3.8) and (3.9), we obtain immediately, since F_n and G_n are disjoint events, that for any $n \in \mathbb{N}$

$$\mathbb{P}_{p+1}(F_n \cup G_n) = \frac{1}{2}\left(\pi + \frac{1}{N-1-p}\right) := \delta > 0. \tag{3.12}$$

Fix $0 < t < T$, and denote by $M = M(t)$ the number of complete excursions occurring before t,

$$M = \max\{n \in \mathbb{N}, \; t_n < t \text{ and } t_{n+1} \geq t\}.$$

By definition of F_n and G_n, we have

$$\{H_t(\partial\Lambda_N) = t\} \subset \cap_{n=0}^M (F_n \cup G_n)^c,$$

so that for any $m \in \mathbb{N}$

$$\mathbb{P}_{p+1}\Big[H_t(\partial\Lambda_N) = t\Big] \leq \mathbb{P}_{p+1}\left(\bigcap_{n=0}^{m}(F_n \cup G_n)^c\right)\mathbb{P}_{p+1}(M \geq m) + \mathbb{P}_{p+1}(M \leq m)$$

$$\leq \mathbb{P}_{p+1}\left(\bigcap_{n=0}^{m}(F_n \cup G_n)^c\right) + \mathbb{P}_{p+1}\left(\max_{0\leq k\leq m} d_k \geq \frac{t}{m+1}\right)$$

$$\leq (1-\delta)^m + (m+1)\mathbb{P}_{p+1}\left(d_0 \geq \frac{t}{m+1}\right)$$

thanks to Eq. (3.12). According to (3.10), $\mathbb{E}_{p+1}(d_0) \leq C/N$. By Markov inequality, the second term is thus less than $C(m+1)^2/tN$. Since $\delta > \pi/2$, we then let $m = -\log N/\log(1-\pi/2)$ to obtain that that for some constant K_1 depending on T, C and π,

$$\mathbb{P}_{p+1}\Big[H_t(\partial\Lambda_N) = t\Big] \leq \frac{K_1(\log N)^2}{tN}. \tag{3.13}$$

Furthermore, using Eqs. (3.8) and (3.9),

$$\mathbb{P}_{p+1}\big[H_t(\partial\Lambda_N) = H(\{N\})\big] \leq \mathbb{P}\big[H(\{N\}) < H(\partial\Lambda_N \setminus \{N\})\big]$$

$$= \frac{1}{2\delta(N-1-p)} \leq \frac{1}{\pi(N-1-p)}. \tag{3.14}$$

Using (3.13), and the bound above, we thus obtain that for any $\varepsilon > 0$, there exists a constant K_2 depending on T, C, π and p, such that

$$\mathbb{P}_{p+1}(H_t(\partial\Lambda_N) = H_t(\{N\})) \leq K_2\frac{N^{\varepsilon-1}}{s},$$

for any $t \in [s, T]$. Letting $K_3 = C(p)K_2$, for any $\varepsilon > 0$, we obtain from Eq. (3.11) that for any $t \in [s, T]$

$$\left|\rho_N(t, p+1) - \sum_{j\in\Lambda_p^*} \alpha_j\mathbb{P}_{p+1}\Big[H(\partial\Lambda_N) = H(\{\mathfrak{d}_j\})\Big|H(\partial\Lambda_N \setminus \{N\}) \leq H(\{N\})\Big]\right|$$

$$\leq K_3\frac{N^{\varepsilon-1}}{s}. \tag{3.15}$$

Let us denote Y a random walk started from Λ_p^*, and driven by the generator $L_{l,N}^\dagger + L_{b,p+1}^\dagger$, where $(L_{b,p+1}^\dagger f)(j)$ is defined in (2.11) as the generator of a symmetric random walk on Λ_p^* with reflection boundary conditions. Denote $\mathbb{Q}_j$ the distribution of Y started from j, and $H^Y(B)$ the hitting time of the set B by Y. Since the only cemetery states that can be reached by Y are the $\mathfrak{d}_j$'s, the Markov property yields

$$\mathbb{P}_{p+1}\Big[H(\partial\Lambda_N) = H(\{\mathfrak{d}_j\})\Big|H(\partial\Lambda_N \setminus \{N\}) \leq H(\{N\})\Big]$$

$$= \mathbb{Q}_p\Big[H^Y(\partial\Lambda_N \setminus \{N\}) = H^Y(\{\mathfrak{d}_j\})\Big]. \tag{3.16}$$

Recall that we denoted μ the unique invariant measure of the generator $L_{l,N} + L_{b,p+1}$, letting

$$\rho^*(j) = \mathbb{E}_\mu(\eta_j)\mathbf{1}_{\{j\in\Lambda_N\}} + \alpha_j\mathbf{1}_{\{j\in\partial\Lambda_N\setminus\{N\}\}},$$

elementary computations similar to those performed for ρ_N yield that the function ρ^* is solution on $\Lambda_p^* \cup \partial \Lambda_N \setminus \{N\}$ of $(L_{l,N}^\dagger + L_{b,p+1}^\dagger)\rho^* = 0$ with boundary condition $\rho^*(\mathfrak{d}_j) = \alpha_j$. By duality, we can therefore write

$$\sum_{j \in \Lambda_p^*} \alpha_j \mathbb{Q}_p \Big[H^Y(\partial \Lambda_N \setminus \{N\}) = H^Y(\{\mathfrak{d}_j\}) \Big] = \mathbb{E}_\mu(\eta_p) = \alpha,$$

where α was introduced in (2.5). Recalling (3.16) then allows (3.15) to be rewritten as wanted

$$\sup_{t \in [s,T]} |\rho_N(t, p+1) - \alpha| \le K_3 \frac{N^{\varepsilon-1}}{s}.$$

We now turn to Eq. (3.4). With only minimal adaptation of the proof above, we can write for a larger constant K_4

$$\sup_{t \in [s,T]} |\rho_N(t, p+2) - \alpha| \le K_4 \frac{N^{\varepsilon-1}}{s}.$$

so that Eq. (3.4) follows immediately from the triangular inequality.

Finally, the third identity (3.5) is proved in the same way as well: we split the random walk started from $N-1$ into excursions away from $N-1$. Each excursion has a probability $1/2$ of ending at site N, where the density is β, and has a probability $1/2(N-1-p)$ of reaching the other boundary. Since the proof is an easier version of that of Eq. (3.3), we do not detail it here. $\qed$

Remark 3.2 (Regarding the assumption $\sum_{j \ne k} a_{j,k} = 0$) The step we just performed is the only point in the proof where we used $\sum_{j \ne k} a_{j,k} = 0$. If the anticopy generator is added, the dual generators must be defined on the set $\{-1, 1\} \times \bar{\lambda}_N$ instead of $\bar{\Lambda}_N$, because one must keep track of the number of times the anticopy generator L_A inverted the value of the site occupied by the random walker. This burdens substantially the notations, therefore we refer the interested reader to [6] for more details on how to overcome this difficulty.

## 3.2 Estimation of the Gradient at the Boundary

Now that we have estimated the density at the boundaries, we estimate the gradients at the boundary, which will be needed later on to estimate the correlations. This is done in Lemma 3.4 below. First, we estimate uniformly in its starting point $k \in \{p+1, \ldots, N-1\}$, the probability that a random walk on $\mathbb{Z}$ reaches either $p+1$ or $N-1$ for the first time in a small time window $[t-s, t]$. Let Y now denote a continuous time random walk in $\mathbb{Z}$ with jump rate N^2 to each neighbor, started from $k \in \{p+1, \ldots, N-1\}$, and driven by the generator $N^2 \Delta_{\mathbb{Z}}$, where for any function $f : \mathbb{Z} \to \mathbb{R}$,

$$(\Delta_{\mathbb{Z}} f)(j) = f(j+1) + f(j-1) - 2f(j).$$

As before, let $H^Y(B)$ be the time at which Y reaches the set $B \subset \mathbb{Z}$,

$$H^Y(B) = \inf\{t \ge 0, \ Y(t) \in B\}.$$

To keep notations simple, we also denote $\mathbb{P}_k$ the distribution of this random walk started at site k.
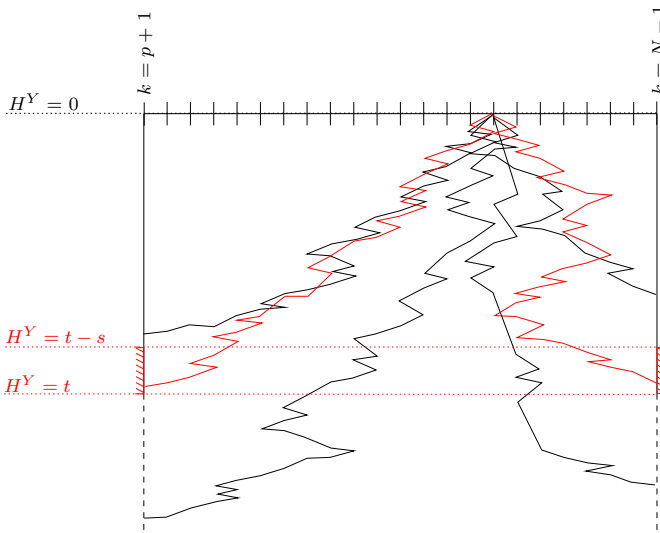
Fig. 1 Lemma 3.3 estimates the probability of the red trajectories uniformly in the starting point of the random walk

Lemma 3.3 *There exists a constant C such that for any* $0 < s < t$

$$\sup_{k \in \{p+1, \ldots, N-1\}} \mathbb{P}_k \left(H^Y(\{p+1, N-1\}) \in [t-s, t] \right) \leq C \left(\frac{s}{t^{3/2}} + \frac{1}{N\sqrt{t-s}} \right).$$

The second term is the error when approximating Y with a Brownian Motion, whereas the first one is the probability above applied to a rescaled Brownian Motion.

Proof of Lemma 3.3 A visual representation of the Lemma is given in Fig. 1. We want to estimate uniformly in $k \in \{p+1, \ldots, N-1\}$ the probability that a random walk started from k hits $p+1$ or $N-1$ between times $t-s$ and t. We first write

$$\mathbb{P}_k \left[H^Y(\{p+1, N-1\}) \in [t-s, t] \right]$$
$$\leq \mathbb{P}_k \left[H^Y(\{N-1\}) \in [t-s, t] \right] + \mathbb{P}_k \left[H^Y(\{p+1\}) \in [t-s, t] \right]$$

The two probabilities on the left hand side are estimated in the same fashion, so that we only estimate the first one. To prove Lemma 3.3, it is therefore sufficient to prove that for some constant C, and any $p+1 \leq k \leq N-1$

$$\mathbb{P}_k \left[H^Y(\{N-1\}) \in [t-s, t] \right] \leq C \left(\frac{s}{t^{3/2}} + \frac{1}{N\sqrt{t-s}} \right) \tag{3.17}$$

Let us denote

$$Z(t) = \max_{t \in [0,T]} Y(t),$$

By reflexion principle

$$\mathbb{P}_k \left[H^Y(\{N-1\}) \in [t-s, t] \right] = \mathbb{P}_k \left[Z(t) \geq N-1 \right] - \mathbb{P}_k \left[Z(t-s) \geq N-1 \right]$$
$$= 2\mathbb{P}_0 \left[Y(t) \geq N-1-k \right] - 2\mathbb{P}_0 \left[Y(t-s) \geq N-1-k \right].$$

Since Y is a random walk sped up by N^2, the family of increments $(Y((k + 1)/N^2) - Y(k/N^2))_{k=0,\dots,tN^2-1}$ is i.i.d. and each of those admits both second and third moments. We can therefore use the Berry–Esseen inequality to write

$$\mathbb{P}_0\Big[Y(t) \geq N - 1 - k\Big] = \mathbb{P}_0\left(\frac{Y(t)}{\sqrt{tN^2}} \geq \frac{N - 1 - k}{\sqrt{tN^2}}\right)$$
$$= 1 - \mathcal{N}\left(\frac{N - 1 - k}{\sqrt{tN^2}}\right) + O\left(\frac{1}{Nt^{1/2}}\right),$$

where $\mathcal{N}(u)$ is the distribution function of a standard Gaussian variable. We can therefore also write

$$\mathbb{P}_0\Big[Y(t - s) \geq N - 1 - k\Big] = 1 - \mathcal{N}\left(\frac{N - 1 - k}{\sqrt{(t - s)N^2}}\right) + O\left(\frac{1}{N\sqrt{t - s}}\right).$$

These two identities allow us to write, since $p + 1 \leq k \leq N - 1$

$$\mathbb{P}_k(H^Y(\{N - 1\}) \in [t - s, t])$$
$$= \mathcal{N}\left(\frac{N - 1 - k}{\sqrt{(t - s)N^2}}\right) - \mathcal{N}\left(\frac{N - 1 - k}{\sqrt{tN^2}}\right) + O\left(\frac{1}{N\sqrt{t - s}}\right)$$
$$\leq \frac{1}{\sqrt{2\pi}}\left(\frac{N - 1 - k}{\sqrt{(t - s)N^2}} - \frac{N - 1 - k}{\sqrt{tN^2}}\right) + O\left(\frac{1}{N\sqrt{t - s}}\right)$$
$$\leq \frac{1}{\sqrt{2\pi t}}\left(\frac{\sqrt{t}}{\sqrt{(t - s)}} - 1\right) + O\left(\frac{1}{N\sqrt{t - s}}\right)$$

One easily obtains after elementary computations a universal constant C such that the first term in the right hand side above is less than $Cs/t^{3/2}$ thus concluding the proof of Lemma 3.3. □

We now use this technical Lemma to prove the following result, which will be needed to estimate the correlations function φ_N.

Lemma 3.4 *There exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, there exists a constant M independent of N, such that*
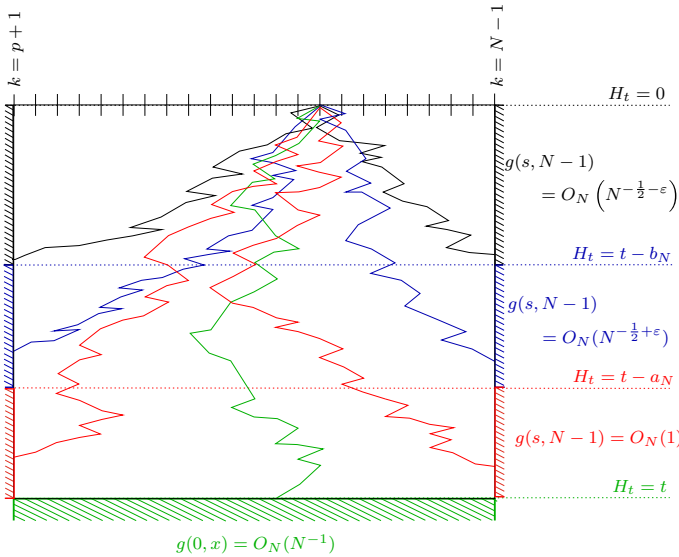
$$\sup_{\substack{t \in [N^{-\varepsilon}, T] \\ k \in \{p+1, \dots, N-1\}}} |\rho_N(t, k + 1) - \rho_N(t, k)| \leq MN^{-\frac{1}{2} - \varepsilon}$$

Proof of Lemma 3.4 This Lemma is a consequence of Proposition 3.1 and Lemma 3.3. For any $k \in \{p + 1, \dots, N - 1\}$, let us denote

$$g(t, k) = \rho_N(t, k + 1) - \rho_N(t, k), \quad \text{and} \quad h(k) = g(0, k) = \rho_0(k + 1/N) - \rho_0(k/N). \tag{3.18}$$

Using Eq. (2.17), we obtain that g is solution to

$$\begin{cases} \partial_t g(t, k) = N^2(\Delta_N g)(t, k) & \forall k \in \{p + 2, \dots, N - 2\} \\ g(t, p + 1) = \rho_N(t, p + 2) - \rho_N(t, p + 1) \\ g(t, N - 1) = \beta - \rho_N(t, N - 1) \\ g(0, .) = h(.) \end{cases} \tag{3.19}$$

Fig. 2 Representation of the four possible cases for the random walk Y

Recall that we denoted by Y a random walk on $\mathbb{Z}$, and that $H^Y(B)$ is the first time Y hits the set $B \subset \mathbb{Z}$, and let $H_t^Y(B) = H^Y(B) \wedge t$. To keep notations simple, shorten

$$H_t = H_t^Y(\{p+1, N-1\}).$$

Then, since g satisfies (3.19), we can write for any $(t, k) \in [0, T] \times \{p+2, \ldots, N-2\}$

$$g(t, k) = \mathbb{E}_k\Big[g(t - H_t, Y(H_t))\Big]. \tag{3.20}$$

According to Proposition 3.1, the more H_t is close to t, the less control we have over the value of g at the boundaries $p + 1$ and $N - 1$. However, the probability that Y reaches either spatial boundary very close to time t is small according to Lemma 3.3. To make this argument rigorous, we fix a small $\delta > 0$, and let

$$a_N = N^{-\frac{1}{2}-\delta} \quad \text{and} \quad b_N = N^{-\frac{1}{2}+\delta}$$

Fix $\delta' > \delta$. We now consider four cases, represented in Fig. 2:

- If $0 \leq H_t \leq t - b_N$, then the random walk has hit the black boundary. Furthermore, thanks to Proposition 3.1, we have a good control of the value of g at the boundary, which is of order $N^{-\frac{1}{2}-\delta'}$.
- If $t - b_N \leq H_t \leq t - a_N$, the random walk hits the blue boundary, where we have some control of the value of g thanks to Proposition 3.1. We also have control over the probability that Y hits the blue boundary thanks to Lemma 3.3, so that the overall contribution of this term is of order N^{-1+c} for some small constant c.
- If $t - a_N \leq H_t < t$, we have no good control over the value of g at the boundary red, which is a priori of order 1. However the probability that Y hits the red boundary is well controlled by Lemma 3.3.

- Finally, if $H_t = t$, the random walk reaches the green boundary (i.e. time 0 for g), and we can write

$$g(t - H_t, Y(H_t)) = h(Y(H_t)) = O(1/N).$$

More precisely, fix a small $\varepsilon > 0$, we can write thanks to Eq. (3.20) for any $t \in [N^{-\varepsilon}, t]$ and any $\delta > 0$

$$
\begin{aligned}
|g(t, k)| &= |\mathbb{E}_k \left(g(t - H_t, Y(H_t)) \right)| \\
&\leq \mathbb{P}_k(0 \leq H_t \leq t - b_N) \sup_{s \in [b_N, t[} |g(s, p+1)| \vee |g(s, N-1)| \\
&\quad + \mathbb{P}_k(t - b_N \leq H_t \leq t - a_N) \sup_{s \in [a_N, b_N]} |g(s, p+1)| \vee |g(s, N-1)| \\
&\quad + \mathbb{P}_k(t - a_N \leq H_t < t) \sup_{s \in [0, a_N]} |g(s, p+1)| \vee |g(s, N-1)| \\
&\quad + \mathbb{P}_k(H_t = t) \sup_{p+2 \leq k \leq N-2} |h(k)| \tag{3.21}
\end{aligned}
$$

We now estimate each of these terms: according to Proposition 3.1, and by definition of the function g, we chose since $\delta' - \delta > 0$, we can write for the first term

$$
\mathbb{P}_k(0 \leq H_t \leq t - b_N) \sup_{s \in [b_N, t[} |g(s, p+1)| \vee |g(s, N-1)|
$$

$$
\leq \sup_{s \in [b_N, t[} |g(s, p+1)| \vee |g(s, N-1)| \leq K \frac{N^{\delta'-\delta-1}}{b_N} \leq K N^{\delta'-2\delta-\frac{1}{2}}. \tag{3.22}
$$

Regarding the second term, we use this time both Proposition 3.1 and Lemma 3.3. For any $t \in [N^{-\varepsilon}, T]$, we let $t' = t - a_N \geq N^{-\varepsilon} - N^{-\frac{1}{2}-\delta}$ and $s = b_N - a_N \leq 2N^{-\frac{1}{2}+\delta}$, to obtain

$$
\mathbb{P}_k(t - b_N \leq H_t \leq t - a_N) \sup_{s \in [a_N, b_N[} |g(s, p+1)| \vee |g(s, N-1)|
$$

$$
\leq C \left(\frac{s}{t'^{3/2}} + \frac{1}{Nt'^{1/2}} \right) K \frac{N^{\delta'-\delta-1}}{a_N} \leq M_1 N^{\frac{3}{2}\varepsilon+\delta'+\delta-1}, \tag{3.23}
$$

for some constant M_1 depending on C and K.

The third term is controlled by Lemma 3.3, and this time we fix $t \in [N^{-\varepsilon}, T]$, and let $s = a_N$, to obtain

$$
\mathbb{P}_k(t - a_N \leq H_t \leq t) \sup_{s \in [0, a_N[} |g(s, p+1)| \vee |g(s, N-1)|
$$

$$
\leq \mathbb{P}_k(t - a_N \leq H_t \leq t) \leq C \left(\frac{s}{t^{3/2}} + \frac{1}{Nt^{1/2}} \right) \leq M_2 N^{\frac{3}{2}\varepsilon-\frac{1}{2}-\delta}. \tag{3.24}
$$

for some constant M_2 depending on C.

Finally, since ρ_0 is was assumed smooth, we also have

$$
\mathbb{P}_k(H_t = t) \sup_{p+2 \leq k \leq N-2} |h(k)| \leq \frac{\|\partial_u \rho_0\|\infty}{N}. \tag{3.25}
$$

We can now choose

$$
\delta = 5\varepsilon/2, \quad \delta' = 4\varepsilon > \delta \text{ and } \varepsilon_0 = 1/18
$$

and inject the four bounds (3.22), (3.23), (3.24) and (3.25) in Eq. (3.21), to finally obtain that for any $0 < \varepsilon < \varepsilon_0$, and any $(t, k) \in [N^{-\varepsilon}, T] \times \{p + 2, \dots, N - 2\}$

$$|g(t, k)| \leq (K + M_1 + M_2 + \|\partial_u \rho_0\|_\infty) N^{-\frac{1}{2} - \varepsilon}.$$

Letting $M = K + M_1 + M_2 + \|\partial_u \rho_0\|_\infty$ then completes the proof of Lemma 3.4. □

The previous estimate yields control over the gradient for macroscopic times of order N^ε, uniformly in Λ_N. We now estimate the gradient of the density for times very close to 0. Since the initial density is not necessarily close to α at the left boundary, and to β at the right boundary, the gradient of the density can be very steep at the boundaries close to the initial time. Away from the boundaries, however, for very small times, the discrete gradient of the density is very close to that of the initial density profile ρ_0, and is therefore of order $1/N$. We now make this statement rigorous.

Lemma 3.5 *Let us denote $x_{N,\varepsilon} = N^{1-\varepsilon/4}$. For any $\varepsilon > 0$, there exists a constant $M' = M'(\varepsilon, \|\partial_u \rho_0\|_\infty)$ such that*

$$\sup_{\substack{t \in [0, N^{-\varepsilon}] \\ k \in \{x_{N,\varepsilon}, \dots, N - x_{N,\varepsilon}\}}} |\rho_N(t, k + 1) - \rho_N(t, k)| \leq \frac{M'}{N}.$$

Proof The proof of this statement also comes from duality. This time, however, the random walk Y is started at a distance at least $x_{N,\varepsilon} = N^{1-\varepsilon/4}$ from the boundary, so that the probability that in a macroscopic time of smaller than $N^{-\varepsilon}$ (i.e. in a microscopic time of order $N^{2-\varepsilon}$), it travels such a distance vanishes exponentially in $N^{\varepsilon/4}$. Once again, we shorten $H_t = H_t^Y(\{p + 1, N - 1\})$. Recall from Eq. (3.18) the definitions of g and h. Since $|g| \leq 1$, following the same steps and using the same notations as in the previous Lemma, we can write for any $k \in \Lambda_N$

$$|g(t, k)| \leq 2\mathbb{P}_k(H_t < t) + \mathbb{P}_k(H_t = t) \sup_{p+2 \leq k \leq N-2} |h(k)| \tag{3.26}$$

$$\leq 2\mathbb{P}_k(H_t < t) + \frac{\|\partial_u \rho_0\|_\infty}{N}. \tag{3.27}$$

As mentioned before, for any $k \in \{x_{N,\varepsilon}, \dots, N - x_{N,\varepsilon}\}$, $\mathbb{P}_k(H_t < t)$ is less than the probability that a rate 1 symmetric random walk travels in a time $\delta t = N^{2-\varepsilon}$ a distance $\delta x = N^{1-\varepsilon/4} = N^{\varepsilon/4}\sqrt{\delta t}$, which can be bounded by $e^{-CN^{\varepsilon/4}}$ for some positive constant C depending only on ε but not on the starting point $k \in \{x_{N,\varepsilon}, \dots, N - x_{N,\varepsilon}\}$, which proves the Lemma. □

Corollary 3.6 *There exists $\varepsilon > 0$ and a constant M_0 independent of N, such that*

$$\sup_{\substack{t \in [0, T] \\ k \in \{x_{N,\varepsilon}, \dots, N - x_{N,\varepsilon}\}}} \left\{ \rho_N(t, k + 1) - \rho_N(t, k) \right\}^2 \leq M_0 N^{-1-2\varepsilon}$$

This corollary is an immediate consequence of Lemmas 3.4 and 3.5, by choosing any $\varepsilon \leq \varepsilon_0$.

4 Estimation of the Correlation Function

4.1 Notations

We now use Lemmas 3.4 and 3.5 to estimate the correlations of the model. The estimation is stated in Proposition 4.1, and uses similar tools as in the previous sections: we obtain a discrete differential system satisfied by the correlation function φ_N, and use duality to estimate φ_N using two-dimensional random walk.

Recall from Eq. (2.9) that we defined the correlation function

$$\varphi_N(t, k, l) = \mathbb{E}_{\nu_N}\Big(\{\eta_k(t) - \rho_N(t, k)\}\{\eta_l(t) - \rho_N(t, l)\}\Big). \tag{4.1}$$

We will denote the two-dimensional equivalents of one-dimensional devices by bolds characters. In particular, we denote pairs of integers by $\boldsymbol{k} = (k, l) \in \mathbb{Z}^2$. For any $\boldsymbol{k} = (k, l) \in \mathbb{Z}^2$, let

$$|\boldsymbol{k}| = |k - l|, \quad \text{and} \quad \|\boldsymbol{k}\| = |k| \vee |l|.$$

For any $\boldsymbol{k} = (k, l)$, we denote $\boldsymbol{\Delta}_N$ the two-dimensional discrete Laplacian

$$
\begin{aligned}
(\boldsymbol{\Delta}_N \varphi_N)(\boldsymbol{k}) &= \sum_{\boldsymbol{k}' \sim \boldsymbol{k}} (\varphi_N(\boldsymbol{k}') - \varphi_N(\boldsymbol{k})) \\
&= \varphi_N(k+1, l) + \varphi_N(k-1, l) + \varphi_N(k, l+1) + \varphi_N(k, l-1) \\
&\quad - 4\varphi_N(k, l),
\end{aligned} \tag{4.2}
$$

and by $\boldsymbol{\nabla}_N \varphi$ the diagonal "gradient"

$$(\boldsymbol{\nabla}_N \varphi)(\boldsymbol{k}) = \varphi_N(k-1, l) + \varphi_N(k, l+1) - 2\varphi_N(k, l). \tag{4.3}$$

For the convenience of notations, we will sometimes write $N^{3/4}$ instead of $\lfloor N^{3/4} \rfloor$.

As represented in Fig. 3, let us introduce the *bulk*

$$B_N = \big\{(k, l), \quad p + 1 < k < N^{3/4} \vee (l - 1), \quad N^{3/4} < l < N\big\},$$

the *diagonal boundary*

$$D_N = \big\{(k, k+1), \quad N^{3/4} \le k \le N - 2\big\},$$

the left *vertical boundary*

$$V_N = \big\{(p+1, l), \quad N^{3/4} < l < N\big\},$$
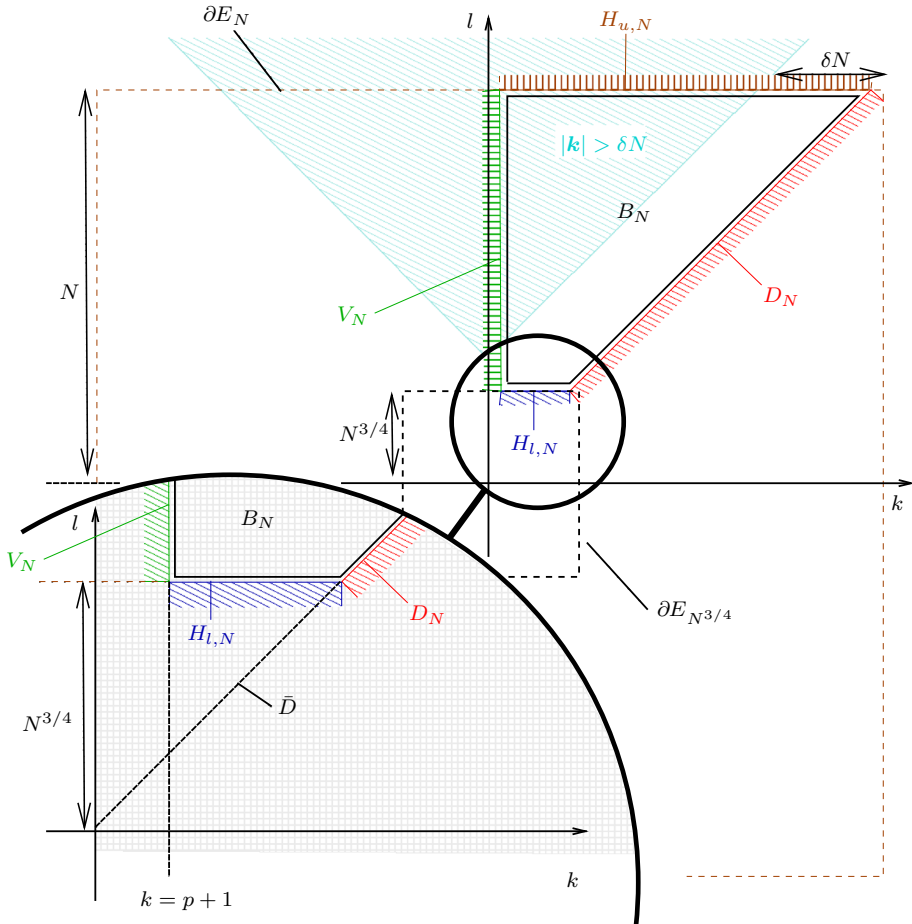
the *lower horizontal border*

$$H_{l,N} = \big\{(k, N^{3/4}), \quad p + 1 \le k < N^{3/4}\big\},$$

and the *upper horizontal border*

$$H_{u,N} = \{(k, N), \quad p + 1 \le k \le N - 2\}.$$

Finally, we denote $\partial B_N = V_N \cup H_{l,N} \cup H_{u,N}$.

Our main result is the following, and states that for any positive δ, at a distance of order δN of both extremities of the diagonal, the correlations vanish uniformly as N goes to ∞. Let us finally shorten $\boldsymbol{0} = (0, 0)$ and $\boldsymbol{N} = (N, N)$.

Fig. 3 Representation of the bulk B_N (black), the diagonal border D_N (red), the vertical border V_N (green), the lower horizontal border $H_{l,N}$ (blue) and the upper horizontal border $H_{u,N}$ (brown) (Color figure online)

Proposition 4.1 *For any $t \in [0, T]$, and any $\delta > 0$*

$$\limsup_{N \to \infty} \sup_{\substack{\boldsymbol{k} \in B_N \\ \|\boldsymbol{k}\|, \|\boldsymbol{k}-\boldsymbol{N}\| > \delta N}} |\varphi_N(t, \boldsymbol{k})| = 0.$$

Proof of Proposition 4.1 For any $(t, \boldsymbol{k}) \in [0, T] \times \boldsymbol{\Lambda}_N$

$$\partial_t \varphi_N(t, \boldsymbol{k}) = N^2 \mathbb{E}_{\nu_N} \Big[L_N \{\eta_k(t) - \rho_N(t, k)\} \{\eta_l(t) - \rho_N(t, l)\} \Big].$$

We will use the notation $\eta_N(t) = \beta = \rho_N(N)$, so that we can extend the definition of φ_N for any time t and any $\boldsymbol{k} = (k, N)$ in the upper boundary $H_{u,N}$, and let

$$\varphi_N(t, \boldsymbol{k}) = 0.$$

With this notation, which defines φ_N at the upper boundary $H_{u,N}$, elementary computations then yield for any $t \in [0, T]$, and any $\boldsymbol{k}$ in B_N

$$\partial_t \varphi_N(t, \boldsymbol{k}) = N^2 (\boldsymbol{\Delta}_N \varphi_N)(t, \boldsymbol{k})$$

where $\boldsymbol{\Delta}_N$ is the discrete two-dimensional Laplacian introduced earlier. We obtain in the same way, for any $\boldsymbol{k} \in D_N$, that

$$\partial_t \varphi_N(t, \boldsymbol{k}) = N^2 (\boldsymbol{\nabla}_N \varphi_N)(t, \boldsymbol{k})) - N^2 m(t, \boldsymbol{k}),$$

where we denoted for $\boldsymbol{k} = (k, k+1) \in D_N$

$$m(t, \boldsymbol{k}) = \left\{ \rho_N(t, k+1) - \rho_N(t, k) \right\}^2,$$

and $\boldsymbol{\nabla}_N$ is the gradient introduced in Eq. (4.3), representing reflection at the diagonal. We do not know yet the value of φ_N neither on the vertical boundary nor on the lower horizontal boundary $H_{l,N}$. However, we already obtained the behavior at the diagonal boundary D_N.

We started our process from a product measure, so that there are no correlations at time 0. By the previous statements, the correlation function φ_N is therefore solution to the discrete difference system

$$\begin{cases} \partial_t \phi(t, \boldsymbol{k}) = N^2 (\boldsymbol{\Delta}_N \phi)(t, \boldsymbol{k}) & \forall (t, \boldsymbol{k}) \in [0, T] \times B_N \\ \partial_t \phi(t, \boldsymbol{k}) = N^2 (\boldsymbol{\nabla}_N \phi)(t, \boldsymbol{k}) - N^2 m(t, \boldsymbol{k}) & \forall (t, \boldsymbol{k}) \in [0, T] \times D_N \\ \phi(t, \boldsymbol{k}) = \varphi_N(t, \boldsymbol{k}) & \forall (t, \boldsymbol{k}) \in [0, T] \times (V_N \cup H_{l,N}) . \quad (4.4) \\ \phi(t, \boldsymbol{k}) = 0 & \forall (t, \boldsymbol{k}) \in [0, T] \times H_{u,N} \\ \phi(0, \boldsymbol{k}) = 0 & \forall \boldsymbol{k} \in B_N \end{cases}$$

Note in particular that the third line gives no informations, but we include it in order to write a discrete difference system with complete boundary conditions. Like we did for the density, we are going to pair φ_N with a random walk X.

4.2 Pairing with a Random Walk

We introduce the infinite diagonal

$$\bar{D} := \{(k, k+1), \quad k \in \mathbb{Z}\}.$$

Note in particular that $D_N \subset \bar{D}$. We denote by X a random walk on $\mathbb{Z}^2$ driven by the generator $N^2 \mathcal{L}$, where for any function $f : \mathbb{Z}^2 \to \mathbb{R}$

$$(\mathcal{L} f)(\boldsymbol{x}) = \mathbf{1}_{\{\boldsymbol{x} \notin \bar{D}\}} (\boldsymbol{\Delta}_N f)(\boldsymbol{x}) + \mathbf{1}_{\{\boldsymbol{x} \in \bar{D}\}} (\boldsymbol{\nabla}_N f)(\boldsymbol{x}). \quad (4.5)$$

In other words, X performs a symmetric random walk in $\mathbb{Z}^2$, and is reflected when hitting $\bar{D}$. We also denote by $\mathbb{P}_{\boldsymbol{k}}$ the distribution of this random walk, started from $\boldsymbol{k}$, and by $\mathbb{E}_{\boldsymbol{k}}$ the corresponding expectation. Similarly to the one-dimensional notations, for any set S, we denote by $\boldsymbol{H}(S)$ the hitting time of S and let $\boldsymbol{H}_t(S) = \boldsymbol{H}(S) \wedge t$. By duality, analogously to the previous section, since φ_N is solution of (4.4), we can then write for any $(t, \boldsymbol{k}) \in [0, T] \times B_N$

$$\varphi_N(t, \boldsymbol{k}) = \mathbb{E}_{\boldsymbol{k}} \left[\varphi_N \left(t - \boldsymbol{H}_t(\partial B_N), X(\boldsymbol{H}_t(\partial B_N)) \right) \right.$$

$$\left. - N^2 \int_{s=0}^{\boldsymbol{H}_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} m(t - s, X(s)) ds \right]. \quad (4.6)$$

Let us denote

$$c_N = \sup_{\substack{t \in [0,T] \\ k \in V_N}} |\varphi_N(t, k)|, \tag{4.7}$$

and note that $|\varphi_N(t, k)| \le 1$ for any t and any k. Equation (4.6) yields that for any $(t, k) \in [0, T] \times B_N$

$$|\varphi_N(t, k)| \le \psi_N(t, k) + N^2 \mathbb{E}_k \left(\int_{s=0}^{H_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} m(t - s, X(s)) ds \right), \tag{4.8}$$

where ψ_N is solution to the system

$$\begin{cases} \partial_t \phi(t, k) = N^2 (\mathbf{\Delta}_N \phi)(t, k) & \forall (t, k) \in [0, T] \times B_N \\ \partial_t \phi(t, k) = N^2 (\mathbf{\nabla}_N \phi)(t, k) & \forall (t, k) \in [0, T] \times D_N \\ \phi(t, k) = c_N & \forall (t, k) \in [0, T] \times V_N \\ \phi(t, k) = 1 & \forall (t, k) \in [0, T] \times H_{l,N} \\ \phi(t, k) = 0 & \forall (t, k) \in [0, T] \times H_{u,N} \\ \phi(0, k) = 0 & \forall k \in B_N \end{cases} . \tag{4.9}$$

The only difference with (4.4) is that we dropped the diagonal increment m, and crudely bounded φ_N by c_N on V_N and by 1 on $H_{l,N}$.

In Corollary 3.6, we obtained control over the value of the increment $m(t, k, k + 1)$ for $t \in [0, T]$ and $k \in \{N^{1-\varepsilon/4}, \dots, N - N^{1-\varepsilon/4}\}$. However, close to the extremities of D_N, m is a priori of order 1, which is an issue due to the factor N^2 in front the increment term. We are therefore going to kill the random walk X when it gets close to either one of the extremities of D_N before $H(\partial B_N)$, and prove that the difference made by doing so is small.

Fix $\varepsilon > 0$ given by Corollary 3.6, we define

$$\widetilde{D}_{N,\varepsilon} = \{(k, k + 1) \in D_N, \quad k \le N^{1-\varepsilon/4} \text{ or } k \ge N - N^{1-\varepsilon/4}\}, \tag{4.10}$$

which is the part of the diagonal D_N where we do not have sufficient control over the diagonal increment m. Shorten

$$H_t^\varepsilon = H_t(\partial B_N \cup \widetilde{D}_{N,\varepsilon}), \tag{4.11}$$

By killing the random walk X at the extremities of the diagonal, we make sure that it does not spend time in the part of D_N where the function m is not well controlled. We can write for this new stopping time

$$\varphi_N(t, k) = \mathbb{E}_k \left(\varphi_N(t - H_t^\varepsilon, X(H_t^\varepsilon)) - N^2 \int_{s=0}^{H_t^\varepsilon} \mathbf{1}_{\{X(s) \in D_N\}} m(t - s, X(s)) ds \right). \tag{4.12}$$

We already pointed out that for any $(t, k) \in [0, T] \times \Lambda_N$,

$$E_k \left[|\varphi_N(t - H_t(\partial B_N), X(H_t(\partial B_N)))| \right] \le \psi_N(t, k),$$

where ψ_N is the solution to (4.9). Since φ_N is bounded in absolute value by 1, the bound above yields

$$\mathbb{E}_k \left(|\varphi_N(t - H_t^\varepsilon, X(H_t^\varepsilon))| \right) \le \psi_N(t, k) + 2\mathbb{P}_k \left[H(\partial B_N) > H(\widetilde{D}_{N,\varepsilon}) \right].$$

Thanks to (4.12), we can therefore write

$$|\varphi_N(t, \boldsymbol{k})| \leq \psi_N(t, \boldsymbol{k}) + 2\mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}(\partial B_N) > \boldsymbol{H}(\widetilde{D}_{N,\varepsilon})\Big]$$

$$+ N^2 \mathbb{E}_{\boldsymbol{k}}\left(\int_{s=0}^{\boldsymbol{H}_t^\varepsilon} \mathbf{1}_{\{X(s)\in D_N\}} m(t-s, X(s)) ds\right).$$

Proposition 4.1 follows from this estimate and Lemmas 4.2, 4.3 and 4.4 below. □

Lemma 4.2 *For any $t \in [0, T]$, and any $\delta > 0$*

$$\limsup_{N\to\infty} \sup_{\substack{\boldsymbol{k}\in B_N \\ \|\boldsymbol{k}\|>\delta N}} \psi_N(t, \boldsymbol{k}) = 0.$$

Lemma 4.3 *For $\varepsilon > 0$ given by Corollary 3.6, and for any $t \in [0, T]$,*

$$\limsup_{N\to\infty} \sup_{\boldsymbol{k}\in B_N} N^2 \mathbb{E}_{\boldsymbol{k}}\left(\int_{s=0}^{\boldsymbol{H}_t^\varepsilon} \mathbf{1}_{\{X(s)\in D_N\}} m(t-s, X(s)) ds\right) = 0. \tag{4.13}$$

Lemma 4.4 *For any $\delta > 0$, and any $\varepsilon > 0$*

$$\limsup_{N\to\infty} \sup_{\substack{(t,\boldsymbol{k})\in[0,T]\times B_N \\ \|\boldsymbol{k}\|, \|\boldsymbol{k}-N\|>\delta N}} \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}(\partial B_N) > \boldsymbol{H}(\widetilde{D}_{N,\varepsilon})\Big] = 0.$$

For the sake of clarity, we prove these three results in separate sections, before completing the proof of Theorem 2.1. To prove these Lemmas, however, the reflected boundary condition at D_N is not convenient. To solve this issue, recall that we defined

$$\bar{D} := \{(k, k+1), \quad k \in \mathbb{Z}\} \supset D_N,$$

we now introduce the symmetry operator $\sigma : \mathbb{Z}^2 \to \mathbb{Z}^2$ w.r.t. $\bar{D}$,

$$\sigma(k, l) = (l - 1, k + 1).$$

We are going to make all the items already introduced symmetric w.r.t. $\bar{D}$. For any set $S \subset \mathbb{Z}^2$ denote

$$S^\sigma = S \cup \sigma S,$$

and for any function f defined on some subset $S \subset \{(k, l) \in \mathbb{Z}^2, \quad k < l\}$ of the half plane above the line D_N, we extend it as a function f^σ on S^σ by symmetry, by letting for any $\boldsymbol{k} \in S$

$$f^\sigma(\sigma \boldsymbol{k}) = f(\boldsymbol{k}).$$

For any $\boldsymbol{k} \in B_N$, we denote by X^σ a random walk on $\mathbb{Z}^2$, started from $\boldsymbol{k}$ and driven by the generator $N^2 \mathcal{L}^\sigma$, where for any function $f : \mathbb{Z}^2 \to \mathbb{R}$

$$(\mathcal{L}^\sigma f)(\mathbf{x}) = \mathbf{1}_{\{x\notin\bar{D}\}}(\boldsymbol{\Delta}_N f)(x) + \frac{1}{2}\mathbf{1}_{\{x\in\bar{D}\}}(\boldsymbol{\Delta}_N f)(x). \tag{4.14}$$

We will denote with exponents σ all the corresponding quantities relative to σ. Note in particular that X^σ is no longer reflected at σ, but it is rather reflected at rate 1/2 and crosses $\bar{D}$ at rate 1/2. With the exception of the time spent on $\bar{D}$, which is double the time spent in any other place, X^σ thus behaves like a rate N^2 continuous time random walk on $\mathbb{Z}^2$. We denote $\boldsymbol{H}^\sigma(S)$ the hitting time of the symmetrized set S^σ by X^σ, and once again

$H_t^\sigma(S) = H^\sigma(S) \wedge t$. The boundary $\partial B_N^\sigma = V_N^\sigma \cup H_{l,N}^\sigma \cup H_{u,N}^\sigma$ is represented in Fig. 4. Further note that we can couple X and X^σ in a way that for any set S contained in the half plane above $\bar{D}$,

$$H(S) = H^\sigma(S^\sigma). \tag{4.15}$$

To build this coupling, given X, one simply has to replace with probability $1/2$, independently, each excursion performed by X away from $\bar{D}$ by its image by the symmetry σ.

We will always assume in what follows that X and X^σ are defined under that coupling, and not to burden the notations, still denote $\mathbb{P}_k$ the corresponding distribution.

4.3 Proof of Lemma 4.2

Before estimating the function ψ_N, we start estimating the correlations between sites $p+1$ and $k > N^{3/4}$ to obtain an upper bound on the quantity c_N defined in (4.7).

Lemma 4.5

$$\limsup_{N\to\infty} c_N = \limsup_{N\to\infty} \sup_{\substack{t\in[0,T]\\ k\in V_N}} |\varphi_N(t,k)| = 0.$$

Before proving this Lemma, we show that it implies Lemma 4.2. Since ψ_N is solution to (4.9), we can write for any $(t,k) \in [0,T] \times B_N$

$$\psi_N(t,k) = \mathbb{E}_k\Big[\psi_N\big(t - H_t(\partial B_N), X(H_t(\partial B_N))\big)\Big].$$

Thanks to Lemma 4.5, ψ_N vanishes uniformly in space and time at the vertical boundary V_N. Furthermore, ψ_N also vanishes at time 0 and at the upper boundary $H_{u,N}$. Therefore the only boundary where ψ_N does not ultimately vanish is $H_{l,N}$. Using the coupling between X and X^σ, and the symmetry identity (4.15), we can write for any t and any $k \in B_N$

$$\begin{aligned}
\psi_N(t,k) &\leq \mathbb{P}_k\Big[H_t(\partial B_N) = H(H_{l,N})\Big] + c_N \mathbb{P}_k\Big[H_t(\partial B_N) = H(V_N)\Big]\\
&\leq \mathbb{P}_k\Big[H_t^\sigma(\partial B_N^\sigma) = H^\sigma(H_{l,N}^\sigma)\Big] + c_N\\
&\leq \mathbb{P}_k\Big[H^\sigma(H_{l,N}^\sigma) < H^\sigma(V_N^\sigma) \vee H^\sigma(H_{u,N}^\sigma)\Big] + c_N\\
&\leq \mathbb{P}_k\Big[H^\sigma(\partial E_{N^{7/8}}) < H^\sigma(\partial E_{2N})\Big] + c_N,
\end{aligned} \tag{4.16}$$

where for any integer K, ∂E_K is the boundary of the box of side $2K$, centered at $\mathbf{0}$

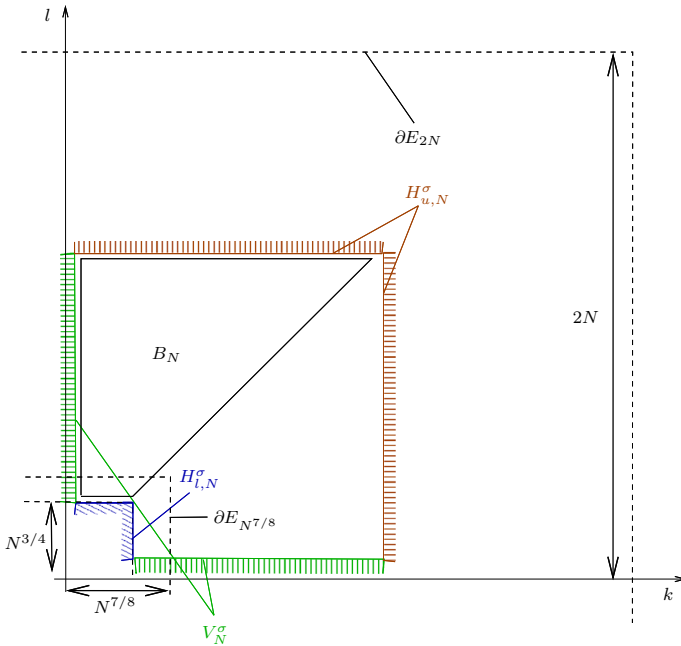$$\partial E_K = \{k \in \mathbb{Z}^2, \quad \|k\| = K\}.$$

The last bound is justified in Fig. 4, where it is shown that if X^σ starts from B_N and if $H(H_{l,N}^\sigma) < H(V_N^\sigma) \vee H(H_{u,N}^\sigma)$, then X^σ reaches $\partial E_{N^{7/8}}$ before ∂E_{2N}.

Furthermore, (cf. Exercise 1.6.8 in [13])

$$\sup_{\substack{k\in B_N\\ \|k\|>\delta_N}} \mathbb{P}_k(H(\partial E_{N^{7/8}}) < H(\partial E_{2N})) \sim \frac{\log 2N - \log \delta N}{\log 2N - \log N^{7/8}} = \frac{8\log\delta}{8\log 2 + \log N}, \tag{4.17}$$

which vanishes as $N \to \infty$ for any fixed δ, as wanted. We now only need to combine Eqs. (4.16) and (4.17), and Lemma 4.5, to prove Lemma 4.2.

We now prove Lemma 4.5.

Fig. 4 Representation of the sets $H_{l,N}^\sigma$, V_N^σ, and $H_{u,N}^\sigma$. Starting from $B_N \cap E_{N^{7/8}}^c$, in order to leave the area delimited by the hatched boundary at $H_{l,N}^\sigma$, a random walk must hit $\partial E_{N^{7/8}}$ before hitting $H_{l,N}^\sigma$ (Color figure online)

Proof of Lemma 4.5 In order to prove this Lemma, consider two random walks X_1 and X_2, respectively started from $p + 1$ and $N^{3/4} < k < N$, and both driven by the dual generator $L_N^\dagger$ defined after (2.13). We are going to prove that before these two particles get close to each other, X_1 will have reached one of the cemetery states $\eth_1, \ldots, \eth_p$ with high probability. Let us denote

$$\bar{D}_p = \{k \in \mathbb{Z}^2, \quad |k| = p + 1\}$$

Let $H(\bar{D}_p)$ be the first time these random walks are at a distance $p + 1$,

$$H(\bar{D}_p) = \inf\{t \geq 0, \quad (X_1(t), X_2(t)) \in \bar{D}_p\}.$$

Let us define $x_{1,N} = N^{3/4}/4 - 1$ and $x_{2,N} = 3N^{3/4}/4 + 1$ and let H_N^1, H_N^2 be defined as

$$H_N^i = \inf\left\{t \geq 0, \quad X_i(s) = x_{i,N}\right\}.$$

Note in particular that in order for $X := (X_1(t), X_2(t)) \in \bar{D}_p$ to occur, either X_1 must have reached $x_{1,N}$ or X_2 must have reached $x_{2,N}$, so that

$$H(\bar{D}_p) > H_N^1 \wedge H_N^2. \tag{4.18}$$

Finally, denote

$$H(\eth) = H^{X_1}(\partial \Lambda_N \setminus \{N\}),$$

the hitting time of one of the cemetery states $\eth_k$ by the first coordinate of the random walk X.

Let us denote by $\widetilde{\mathbb{P}}_k$ the joint distribution of X_1, X_2, moving independently, where the first is started from $p + 1$ and the second from $k \geq N^{3/4}$. We claim that X_1 reaches one of the cemetery states before the two random walks get close to each other, or

$$\limsup_{N \to \infty} \ \sup_{N^{3/4} \leq k \leq N} \ \widetilde{\mathbb{P}}_k \Big[H(\mathfrak{d}) > H(\bar{D}_p) \Big] = 0. \tag{4.19}$$

Denote $s_N = N^{-3/4}$. For any $N^{3/4} \leq k \leq N$, Eq. (4.18) yields

$$\widetilde{\mathbb{P}}_k \Big[H(\mathfrak{d}) > H(\bar{D}_p) \Big] \leq \widetilde{\mathbb{P}}_k \Big[H(\mathfrak{d}) \geq s_N \Big] + \widetilde{\mathbb{P}}_k \Big[H(\mathfrak{d}) \geq H_N^1 \Big] + \widetilde{\mathbb{P}}_k \Big[H_N^2 \leq s_N \Big]. \tag{4.20}$$

Since X_1 and X_2 behave as random walkers until they are in Λ_p^*, the last term is less than the probability that a rate N^2 symmetric random walk on $\mathbb{Z}$ travels a distance of order $N^{3/4}$ before time s_N. However, because of the acceleration in N^2, in a time s_N, X_2 would typically travel a distance of order $\sqrt{s_N N^2} = N^{5/8} = N^{3/4} N^{-1/8}$. Elementary computations and a large deviations estimate therefore yields that for some constant C independent of N and $k \geq N^{3/4}$

$$\widetilde{\mathbb{P}}_k(H_N^2 \leq s_N) \leq e^{-C N^{1/8}}.$$

Furthermore, using minimal adaptations of Eqs. (3.13) and (3.14), we obtain

$$\widetilde{\mathbb{P}}_k(H(\mathfrak{d}) \geq s_N) \leq \frac{K (\log N)^2}{s_N N},$$

where K is a constant depending on C and π, and

$$\widetilde{\mathbb{P}}_k(H(\mathfrak{d}) \geq H_N^1) \leq \frac{2}{\pi (x_{1,N} - 1 - p)}.$$

These three bounds and Eq. (4.20) yield that for any $N^{3/4} \leq k \leq N - 1$, $\varepsilon < 1/4$, and N large enough,

$$\widetilde{\mathbb{P}}_k \Big[H(\mathfrak{d}) > H(\bar{D}_p) \Big] \leq N^{-\varepsilon},$$

which proves Eq. (4.19).

We now get back to estimating the correlations and proving Lemma 4.5. In order not to introduce burdensome notations, we will not write in full detail this part of the proof, which relies once again on duality. As we did to estimate the density, we pair φ_N with a two-dimensional random walk $X = (X_1, X_2)$ on $\bar{\Lambda}_N^2$. Let us shorten

$$H = H(\mathfrak{d}) \wedge H(\bar{D}_p) \quad \text{and} \quad H_t = H \wedge t.$$

If X reaches either time t or one of the cemetery states $(\mathfrak{d}_k, l)$, with $l > p + 1$ not in the boundary, φ_N vanishes. Furthermore, since before τ, we have $|X_1 - X_2| > p + 1$, X_1 and X_2 are distributed at least until τ as independent random walks with generator $N^2 L_N^\dagger$. We can therefore write, noting that X cannot reach the diagonal $\bar{D}$ before time τ, and since $|\varphi_N|$ is less than 1 and vanishes at time 0,

$$\begin{aligned} \varphi_N(t, p+1, k) &= \mathbb{E}_{(p+1,k)}(\varphi_N(t - H_t, X(H_t))) \\ &\leq \widetilde{\mathbb{P}}_k(H(\mathfrak{d}) > H(\bar{D}_p)) + \mathbb{E}_{(p+1,k)}(\underbrace{\varphi_N(t - H, X(H)) \mathbf{1}_{\{H(\mathfrak{d}) \leq H(\bar{D}_p)\}}}_{=0}), \end{aligned}$$

so that (4.19) concludes the proof of Lemma 4.5. $\qquad \square$

4.4 Proof of Lemma 4.3

We now estimate the overall contribution of the diagonal increments to φ_N. Recall that we want to estimate

$$N^2 \mathbb{E}_k \left(\int_{s=0}^{H_t^\varepsilon} \mathbf{1}_{\{X(s) \in D_N\}} m(t-s, X(s)) ds \right),$$

where we shortened $H_t^\varepsilon = H_t(\partial B_N \cup \widetilde{D}_{N,\varepsilon})$ and $\widetilde{D}_{N,\varepsilon}$ is the set of points -defined in (4.10)- of the diagonal D_N at distance at most $N^{1-\varepsilon/4}$ of its extremities. By definition of H_t^ε, for any $s \in [0, H_t^\varepsilon)$, we cannot have $X \in \widetilde{D}_{N,\varepsilon}$. In particular, for $\varepsilon > 0$ given by Corollary 3.6, $k \in B_N$,

$$N^2 \mathbb{E}_k \left(\int_{s=0}^{H_t^\varepsilon} \mathbf{1}_{\{X(s) \in D_N\}} m(t-s, X(s)) ds \right)$$

$$\leq M_0 N^{1-2\varepsilon} \mathbb{E}_k \left(\int_{s=0}^{H_t^\varepsilon} \mathbf{1}_{\{X(s) \in D_N\}} ds \right).$$

Now that the problem of controlling m is dealt with, we can get back to the real stopping time $H_t(\partial B_N)$ (which is by definition larger than H_t^ε) and write

$$N^2 \mathbb{E}_k \left(\int_{s=0}^{H_t^\varepsilon} \mathbf{1}_{\{X(s) \in D_N\}} m(t-s, X(s)) ds \right)$$

$$\leq M_0 N^{1-2\varepsilon} \mathbb{E}_k \left(\int_{s=0}^{H_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} ds \right). \tag{4.21}$$

In order to simplify the problem, we start by making it symmetric w.r.t. the line $\bar{D} = \{(k, k+1), \quad k \in \mathbb{Z}\}$. To do so, we use once again the random walk X^σ introduced earlier, with generator given by (4.14). By construction, X and X^σ spend the same time in D_N, therefore

$$\int_{s=0}^{H_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} ds = \int_{s=0}^{H_t^\sigma(\partial B_N^\sigma)} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds,$$

where $H_t^\sigma(\partial B_N^\sigma)$ was introduced just before (4.15). Recall that ∂E_{2N} is the set of vertices k such that $\|k\| = 2N$, and that $H(\partial E_{2N})$ is the first time X hits the boundary ∂E_{2N}. Assuming that X^σ starts in B_N, we can write according to Fig. 4 that $H_t^\sigma(\partial B_N^\sigma) \leq H^\sigma(\partial B_N^\sigma) < H^\sigma(\partial E_{2N})$, so that

$$\int_{s=0}^{H_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} ds \leq \int_{s=0}^{H^\sigma(\partial E_{2N})} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds.$$

This last bound and Eq. (4.21) finally yield that for any N large enough

$$N^2 \mathbb{E}_k \left(\int_{s=0}^{H_t(\partial B_N)} \mathbf{1}_{\{X(s) \in D_N\}} m(t-s, X(s)) ds \right)$$

$$\leq M_0 N^{1-2\varepsilon} \mathbb{E}_k \left(\int_{s=0}^{H^\sigma(\partial E_{2N})} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds \right),$$

therefore Lemma 4.3 follows from Lemma 4.6 below.

Lemma 4.6 *For any $c > 0$,*

$$\limsup_{N \to \infty} \sup_{\boldsymbol{k} \in B_N} N^{1-c} \mathbb{E}_{\boldsymbol{k}} \left(\int_{s=0}^{\boldsymbol{H}^\sigma(\partial E_{2N})} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds \right) = 0.$$

Proof of Lemma 4.6 In order to simplify the problem, we introduce a discrete time random walk $(\boldsymbol{Z}_m)_{m \geq 0}$ on $\mathbb{Z}^2$, started from $\boldsymbol{k}$ as well, and performing the exact same jumps as X^σ. Then, shortening $\boldsymbol{H}^Z := \boldsymbol{H}^Z(\partial E_{2N})$ the (discrete) time at which $\boldsymbol{Z}$ reaches the boundary ∂E_{2N}, and since the waiting time of X^σ at any site in $D_N \subset \bar{D}$ has distribution $Exp(2N^2)$ (4 neighbors, each jumped to at rate $N^2/2$), we can write

$$\mathbb{E}_{\boldsymbol{k}} \left(\int_{s=0}^{\boldsymbol{H}^\sigma(\partial E_{2N})} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds \right) = \frac{1}{2N^2} \mathbb{E}_{\boldsymbol{k}} \left(\sum_{m=1}^{\boldsymbol{H}^Z} \mathbf{1}_{\{Z_m \in D_N\}} \right). \tag{4.22}$$

Recall that $E_{2N} = \{\boldsymbol{k} \in \mathbb{Z}^2, \quad \|\boldsymbol{k}\| < 2N\}$ is the discrete box of size $2N$. Fix some $\boldsymbol{k}, \boldsymbol{k}' \in E_{2N}$, we now compute

$$\psi_{\boldsymbol{k}}(\boldsymbol{k}') := \mathbb{E}_{\boldsymbol{k}} \left(\sum_{m=1}^{\boldsymbol{H}^Z} \mathbf{1}_{\{Z_m = \boldsymbol{k}'\}} \right).$$

Since $\boldsymbol{Z}$ performs a symmetric random walk until reaching ∂E_{2N}, $\psi_{\boldsymbol{k}}$ is solution to

$$\begin{cases} (\boldsymbol{\Delta}_N \phi(\boldsymbol{k}') = 0 & \forall \boldsymbol{k}' \in E_{2N} \setminus \{\boldsymbol{k}\} \\ \phi(\boldsymbol{k}') = 0 & \forall \boldsymbol{k}' \in \partial E_{2N} \\ \phi(\boldsymbol{k}) = 1/p_{\boldsymbol{k},N} \end{cases},$$

where $p_{\boldsymbol{k},N}$ is the probability for $\boldsymbol{Z}$, starting from $\boldsymbol{k}$ to reach ∂E_{2N} without coming back to $\boldsymbol{k}$, which is also, starting from $\boldsymbol{k}$, the expectation of the number of passages in $\boldsymbol{k}$ before reaching the boundary ∂E_{2N}. By maximum principle, we can now crudely bound $\psi_{\boldsymbol{k}}(\boldsymbol{k}')$, uniformly in $\boldsymbol{k}'$, by $1/p_{\boldsymbol{k},N}$, so that the right hand side in (4.22) is bounded from above for any $\boldsymbol{k}$ by

$$\frac{1}{2N^2} \sum_{\boldsymbol{k}' \in D_N} \psi_{\boldsymbol{k}}(\boldsymbol{k}') \leq \frac{\#D_N}{2N^2 p_{\boldsymbol{k},N}} \leq \frac{1}{2N p_{\boldsymbol{0},N}}.$$

The last holds due to the probability to leave $\boldsymbol{k}$ and never come back before reaching the boundary ∂E_{2N} being smallest for $\boldsymbol{k} = \boldsymbol{0}$, and because the cardinal of D_N is $N - 2 \leq N$. As N goes to infinity, we have $p_{\boldsymbol{0},N} \geq K/\log N$, so that for any N large enough,

$$\sup_{\boldsymbol{k} \in B_N} N^{1-c} \mathbb{E}_{\boldsymbol{k}} \left(\int_{s=0}^{\boldsymbol{H}^\sigma(\partial E_{2N})} \mathbf{1}_{\{X^\sigma(s) \in D_N\}} ds \right) \leq \frac{\log N}{2K N^c},$$

for some fixed constant K. This concludes the proof of the Lemma. □

4.5 Proof of Lemma 4.4

Recall that ε and δ are fixed, small, positive constants, that

$$\widetilde{D}_{N,\varepsilon} = \{(k, k+1) \in D_N, \quad k \leq N^{1-\varepsilon/4} \text{ or } k \geq N - N^{1-\varepsilon/4}\}. \tag{4.23}$$

and that we want to prove that $\mathbb{P}_{\boldsymbol{k}} \left[\boldsymbol{H}(\partial B_N) > \boldsymbol{H}(\widetilde{D}_{N,\varepsilon}) \right]$ vanishes, as $N \to \infty$, uniformly in $t \in [0, T]$ and $\boldsymbol{k} \in B_N$ such that $\|\boldsymbol{k}\|, \|\boldsymbol{N} - \boldsymbol{k}\| > \delta N$.

Once again, let us make our problem symmetric w.r.t the diagonal D_N, and recalling the notations introduced after Lemma 4.4, write

$$\mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}(\partial B_N) > \boldsymbol{H}(\widetilde{D}_{N,\varepsilon})\Big] = \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(\partial B_N^\sigma) > \boldsymbol{H}^\sigma(\widetilde{D}_{N,\varepsilon})\Big].$$

(Of course, since $\widetilde{D}_{N,\varepsilon} \subset \bar{D}$, we have $\widetilde{D}_{N,\varepsilon}^\sigma = \widetilde{D}_{N,\varepsilon}$) For any $\boldsymbol{k}, \ell \in \mathbb{Z}^2 \times \mathbb{N}$, let us denote

$$E_\ell(\boldsymbol{k}) = \{\boldsymbol{k}' \in \mathbb{Z}^2, \|\boldsymbol{k} - \boldsymbol{k}'\| \le \ell\},$$
$$\partial E_\ell(\boldsymbol{k}) = \{\boldsymbol{k}' \in \mathbb{Z}^2, \|\boldsymbol{k} - \boldsymbol{k}'\| = \ell\}.$$

Then, letting $\ell_N = 2N^{1-\varepsilon/4}$, we have

$$\widetilde{D}_{N,\varepsilon} \subset E_{\ell_N}(\boldsymbol{0}) \cup E_{\ell_N}(\boldsymbol{N}),$$

therefore by union bound,

$$\mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}(\partial B_N) > \boldsymbol{H}(\widetilde{D}_{N,\varepsilon})\Big]$$
$$\le \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{0})) < \boldsymbol{H}^\sigma(\partial B_N^\sigma)\Big] + \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{N})) < \boldsymbol{H}^\sigma(\partial B_N^\sigma)\Big]. \quad (4.24)$$

Since we assume both $\|\boldsymbol{k}\|$ and $\|\boldsymbol{N} - \boldsymbol{k}\|$ to be larger than δN, both of the probabilities on the right hand side are estimated in the same way, so that we will only estimate the first one. To do so, simply note that for any $\boldsymbol{k} \in B_N$, $\boldsymbol{H}^\sigma(\partial B_N^\sigma) \le \boldsymbol{H}^\sigma(\partial E_{2N})$, so that

$$\mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{0})) < \boldsymbol{H}^\sigma(\partial B_N^\sigma)\Big] \le \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{0})) < \boldsymbol{H}^\sigma(\partial E_{2N}^\sigma(\boldsymbol{0}))\Big].$$

The left hand side above can be written as

$$\frac{\log(\|\boldsymbol{k}\|) - \log(2N)}{\log(\ell_N) - \log(2N)} + o_N(1),$$

where the $o_N(1)$ vanishes uniformly in $\boldsymbol{k}$. In particular,

$$\sup_{\substack{\boldsymbol{k} \in B_N \\ \|\boldsymbol{k}\| > \delta N}} \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{0})) < \boldsymbol{H}^\sigma(\partial B_N^\sigma)\Big] \le \frac{\varepsilon}{4}\left(\frac{\log 2 - \log \delta}{\log N}\right).$$

We obtain similarly

$$\sup_{\substack{\boldsymbol{k} \in B_N \\ \|\boldsymbol{N} - \boldsymbol{k}\| > \delta N}} \mathbb{P}_{\boldsymbol{k}}\Big[\boldsymbol{H}^\sigma(E_{\ell_N}(\boldsymbol{N})) < \boldsymbol{H}^\sigma(\partial B_N^\sigma)\Big] \le \frac{\varepsilon}{4}\left(\frac{\log 2 - \log \delta}{\log N}\right).$$

Together with (4.24), these two bounds conclude the proof of Lemma 4.4.

5 Proof of Theorem 2.1

We now have all the tools needed to prove the hydrodynamic limit. Fix a continuous function $G : [0, 1] \to \mathbb{R}$, and $t \in [0, T]$. Then, Using triangular and Cauchy Schwarz inequalities,

we can estimate the square of the quantity inside the expectation in Theorem 2.1 by

$$\left(\frac{1}{N} \sum_{k \in \Lambda_N} G(k/N) \left[\eta_k(t) - \bar{\rho}(t, k/N) \right] \right)^2$$

$$\leq \frac{C(p)\|G\|_\infty}{N} + \frac{2}{N^2} \left(\sum_{k=p+2}^{N-1} G(k/N) \left[\eta_k(t) - \rho_N(t, k) \right] \right)^2$$

$$+ \frac{2}{N^2} \left(\sum_{k=p+2}^{N-1} G(k/N) \left[\rho_N(t, k) - \bar{\rho}(t, k/N) \right] \right)^2$$

$$\leq \frac{C(p)\|G\|_\infty}{N} + \frac{2}{N^2} \sum_{k,l=p+2}^{N-1} G(k/N) G(l/N) \varphi_N(t, k, l)$$

$$+ 2\|G\|_\infty^2 \frac{1}{N} \sum_{k=p+2}^{N-1} \left[\rho_N(t, k) - \bar{\rho}(t, k/N) \right]^2.$$

For any positive δ, the first sum on the right-hand side is less than

$$\sum_{\substack{k \in B_N \\ \|k\|, \|N-k\| > \delta N}} \frac{2\|G\|_\infty^2 |\varphi_N(t, \boldsymbol{k})|}{N^2}$$

$$+ \frac{2\|G\|_\infty^2}{N^2} \# \Big\{ \boldsymbol{k} \in \{p+1, \ldots, N-1\}^2 \quad \|\boldsymbol{k}\| \wedge \|\boldsymbol{N} - \boldsymbol{k}\| \leq \delta N \Big\}.$$

The first term vanishes as $N \to \infty$ for any $\delta > 0$ according to Proposition 4.1, whereas the second converges as $N \to \infty$ to $C\delta^2$ for some constant C. We then let $\delta \to 0$, so that Theorem 2.1 follows from Lemma 5.1 below.

Lemma 5.1 *For any $t \in [0, T]$*

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(t, k) - \bar{\rho}(t, k/N))^2 = 0.$$

Proof of Lemma 5.1 To estimate the quantity above, we compute its time derivative

$$\partial_t \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(t, k) - \bar{\rho}(t, k/N))^2$$

$$= \frac{2}{N} \sum_{k=p+2}^{N-1} (\rho_N(t, k) - \bar{\rho}(t, k/N))(N^2(\Delta_N \rho_N)(t, k) - (\Delta\bar{\rho})(t, k/N)).$$

Note that because the boundary conditions are not a priori respected by the initial profile ρ_0, the space derivative of $\bar{\rho}$ can diverge as $t \to 0$. However, Since $\bar{\rho}$ is smooth, for any $\varepsilon > 0$, uniformly in $t \in [\varepsilon, T]$, we can write

$$(\Delta\bar{\rho})(t, k/N) = N^2(\bar{\rho}(t, (k+1)/N) + \bar{\rho}(t, (k-1)/N) - 2\bar{\rho}(t, k/N)) + C(\varepsilon)o_N(1),$$

where $C(\varepsilon)$ can diverge as $\varepsilon \to 0$. For any $k \in [\![-1, N]\!]$, let us denote

$$\theta(t, k) = \rho_N(t, k) - \bar{\rho}(t, k/N).$$

Thanks to the identity above, for any $t \in [\varepsilon, T]$ we obtain by integration by parts

$$\partial_t \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(t,k) - \bar{\rho}(t,k/N))^2 = 2N \sum_{k=p+2}^{N-1} \theta(t,k)(\Delta_N \theta)(t,k) + C(\varepsilon)o_N(1)$$

$$= -2N \left[\sum_{k=p+1}^{N} \{\theta(t,k+1) - \theta(t,k)\}^2 + \theta(t,N)\{\theta(t,N) - \theta(t,N-1)\} \right.$$

$$\left. - \theta(t,p+1)\{\theta(t,p+2) - \theta(t,p+1)\} \right] + C(\varepsilon)o_N(1)$$

The first sum above is negative, and does therefore not need to be controlled. Furthermore, for any $\varepsilon > 0$, the last term vanishes as $N \to \infty$. We now take a look at the two other terms. They are treated in the same way, so that we only consider the second. The general idea is that θ is at most $O(N^{-\frac{1}{2}})$, whereas the second factor is a gradient of order at least $o(N^{-\frac{1}{2}-\delta})$. More precisely,

$$N^{\frac{1}{2}} |\theta(t,p+1)| = N^{\frac{1}{2}} |\rho_N(t,p+1) - \bar{\rho}(t,(p+1)/N)|$$

$$\leq N^{\frac{1}{2}} |\rho_N(t,p+1) - \alpha| + N^{\frac{1}{2}} |\alpha - \bar{\rho}(t,(p+1)/N)|.$$

For any $\varepsilon > 0$, the first term vanishes uniformly in $t \in [\varepsilon, T]$ according to Proposition 3.1. The second term vanishes as well, uniformly in $t \in [\varepsilon, T]$, because $\bar{\rho}$ is smooth. Similarly, for any $\delta > 0$,

$$N^{\frac{1}{2}+\delta} |\theta(t,p+2) - \theta(t,p+1)| \leq N^{\frac{1}{2}+\delta} |\rho_N(t,p+2) - \rho_N(t,p+1)|$$

$$+ N^{\frac{1}{2}+\delta} |\bar{\rho}(t,(p+2)/N) - \bar{\rho}(t,(p+1)/N)|.$$

Once again, both terms vanish uniformly in $t \in [\varepsilon, T]$ according to Proposition 3.1 and because $\bar{\rho}$ is smooth. The term $N\theta(t,N)(\theta(t,N) - \theta(t,N-1))$ is estimated in the same fashion.

Finally, for any ε, and any $t \in [\varepsilon, T]$,

$$\partial_t \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(t,k) - \bar{\rho}(t,k/N))^2$$

$$\leq -2N \sum_{k=p+1}^{N} (\theta(t,k+1) - \theta(t,k))^2 + C(\varepsilon)o_N(1).$$

We can thus write, for any $\varepsilon > 0$ and $t \in [\varepsilon, T]$

$$\frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(t,k) - \bar{\rho}(t,k/N))^2 \leq \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(\varepsilon,k) - \bar{\rho}(\varepsilon,k/N))^2 + C(\varepsilon)o_N(1).$$

Lemma 5.1 therefore follows from Lemma 5.2 below. □

Lemma 5.2

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(\varepsilon,k) - \bar{\rho}(\varepsilon,k/N))^2 = 0.$$

Proof of Lemma 5.2 For any $k \in \{p + 1, N - 1\}$ (resp. $u \in [0, 1]$) let $\mathbb{P}_k$ (resp. $\widetilde{\mathbb{P}}_u$) be the distribution of a continuous time random walk X on $\mathbb{Z}$ (resp. a standard Brownian motion B) started from k (resp. from u), and jumping at rate N^2 to any of its neighbors. Let $\mathbb{E}_k$ (resp. $\widetilde{\mathbb{E}}_u$) denote the corresponding expectation. Fix $k \in \{p + 2, N - 1\}$, we write for any ε

$$\rho_N(\varepsilon, k) = \mathbb{E}_k \Big[\rho_N(\varepsilon - H_\varepsilon, X(H_\varepsilon)) \Big] \tag{5.1}$$

where we shortened $H_\varepsilon := H(\{p + 1, N - 1\}) \wedge \varepsilon$ and $H(\{p + 1, N - 1\})$ is X's hitting time of the boundary $\{p + 1, N - 1\}$. Similarly,

$$\bar{\rho}(\varepsilon, k/N) = \widetilde{\mathbb{E}}_{k/N} \Big[\rho_N(\varepsilon - \widetilde{H}_\varepsilon, B(\widetilde{H}_\varepsilon)) \Big], \tag{5.2}$$

where $\widetilde{H}_\varepsilon = \widetilde{H}(\{0, 1\}) \wedge \varepsilon$ and $\widetilde{H}(\{0, 1\})$ is B's hitting time of the boundary $\{0, 1\}$.

We now use both of these identities to prove that, at distance at least $\varepsilon^{1/4} N$ from the boundary, the density is close to its initial value. Fix k such that

$$p + 2 + \varepsilon^{1/4} N < k < N - 1 - \varepsilon^{1/4} N.$$

Then, we obtain from (5.1), since ρ_N is bounded in absolute value by 1,

$$|\rho_N(\varepsilon, k) - \rho_0(k/N)| \leq 2 \mathbb{P}_k \left(|X(H_\varepsilon) - k| \geq \varepsilon^{1/4} N \right)$$
$$+ \sup_{|k - k'| \leq \varepsilon^{1/4} N} \left| \rho_0(k/N) - \rho_0(k'/N) \right|.$$

In a time ε, X would typically travel a distance $\sqrt{\varepsilon} N$, so that the first term is $O(e^{-\varepsilon^{-1/4}})$. Since ρ_0 is smooth, the second term is $O(\varepsilon^{1/4})$. Using this time Eq. (5.2), we can write an analogous bound for $\bar{\rho}$ so that for any $\varepsilon^{1/4} \leq u \leq 1 - \varepsilon^{1/4}$

$$|\bar{\rho}(\varepsilon, u) - \rho_0(u)| \leq 2 \widetilde{\mathbb{P}}_u \left(|B(\widetilde{H}_\varepsilon) - u| \geq \varepsilon^{1/4} \right) + \sup_{|u - u'| \leq \varepsilon^{1/4}} \left| \rho_0(u) - \rho_0(u') \right|$$
$$= O_\varepsilon(e^{-\varepsilon^{-1/4}}) + O_\varepsilon(\varepsilon^{1/4}).$$

We finally obtain for any $p + 2 + \varepsilon^{1/4} N < k < N - 1 - \varepsilon^{1/4} N$

$$|\rho_N(\varepsilon, k) - \bar{\rho}(\varepsilon, k/N)| = o_\varepsilon(\varepsilon),$$

where the $o_\varepsilon(1)$ is uniform in k and can be chosen independent of N.

Since for any k,

$$|\rho_N(\varepsilon, k) - \bar{\rho}(\varepsilon, k/N)| \leq 1,$$

we can now estimate

$$\frac{1}{N} \sum_{k=p+2}^{N-1} (\rho_N(\varepsilon, k) - \bar{\rho}(\varepsilon, k/N))^2$$

$$\leq \frac{1}{N} \sum_{k=p+2+\varepsilon^{1/4}N}^{N-1-\varepsilon^{1/4}N} (\rho_N(\varepsilon, k) - \bar{\rho}(\varepsilon, k/N))^2 + \frac{p+2}{N} + 2\varepsilon^{1/4} = O_\varepsilon(1) + O_N(1),$$

where $O_\varepsilon(1)$ does not depend on N, which proves Lemma 5.2. $\qquad \square$

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