KPZ Universality conjectures and KPZ universality class

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1 Introduction : Random and ballistic deposition

So far, all the models studied linked to the KPZ univerality class are in $(1+1)$ *dimensions (one space& one time dimension)*

- Blocks drop at rate one on every site of $\mathbb{Z}$
- First case : no interactions between the blocks, independant heights $(h_z)_{z\in\mathbb{Z}}$: *random deposition*
- Second case, the blocks stick to their neighbors, heights are no longer independant

The expected scales for the fluctuation of the depositions models are

Random deposition

- Height expectation $\sim O(t)$
- Height fluctuations $\sim O(t^{1/2})$
- No spatial correlation

Ballistic deposition

- Height expectation $\sim O(t)$
- Height fluctuations $\sim O(t^{1/3})$
- Spatial correlations $\sim O(t^{2/3})$

The ballistic deposition model has three main characteristics :

KPZ growth characteristics

1. *smoothing* : the heights tend to homogeneize
2. *slope dependant growth* : when the slope is large, growth occurs more quickly
3. *space time uncorrelated noise :* independant blocks fall

2 Universality classes

2.1 Gaussian universality class :

CLT : By understanding normal varibales, the CLT gives intel on any average of random variable.

- By studying one object (BM/Gaussian variable), one can obtain results on a wide variety of models and quantities.
- The other way around, by showing things on discrete models, obtain results on the continuum limit

2.2 KPZ universality class :

Any growth model with these characteristics is expected to be in the KPZ universality class

$\mapsto$ Main characteristics of the KPZ universality class is **fluctuations** of order $t^{1/3}$, and **spatial correlations** of order $t^{2/3}$.

Despite significant progress in the last decades, up to this point, the KPZ universality is not fully understood. (Cf. Universality conjectures, later in the talk)

3 The KPZ equation

General form of the KPZ equation

$$\boxed{\partial_T h = \nu \partial_X^2 h + \lambda (\partial_X h)^2 + \sigma \dot{W}}$$

- 3 parameters ν, λ and σ.
- By space and time rescaling, one can drop two of the parameters.
- KPZ equation $\rightarrow$ KPZ equation<u>s</u>, the $''KPZ$ equation" is in fact a one parameter family KPZ(γ).

1) $\partial_X^2 h$: smoothing out 2) $(\partial_X h)^2$ Slope-dependent growth 3) $\dot{W}$ space-time white noise

It is expected that the KPZ universality involves both the scaling exponents as well as the **long-time distributions**, within **geometry-dependent subclasses**. These geometry-dependent subclasses depend on the initial profile. To illustrate that, ASEP.

4 The Asymmetric Simple Exclusion Process (ASEP)

4.1 The asymmetric exclusion process

Description of the model :

- On $\mathbb{Z}$, each site is either occupied ($\eta_x = 1$) or empty ($\eta_x = 0$)
- fix a (possibly random) initial configuration
- Each particles moves from x to $x + 1$ at rate $1/2 < p < 1$

- Each particles moves from x to $x-1$ at rate $q = 1 - p$
- *Exclusion rule* : any motion towards an occupied site is cancelled

4.2 Various cases for the asymmetry

We denote by $\gamma = p - q$ the asymmetry of the system, therefore $p = (1+\gamma)/2$, and $q = (1-\gamma)/2$.

1. $\gamma = 0$, Symmetric Simple Exclusion Process (SSEP)
2. $\gamma = 1$, Totally Asymmetric Simple Exclusion Process (TASEP)
3. $0 < \gamma < 1$, Partially Asymmetric Simple Exclusion Process (PASEP)
4. $\gamma = \varepsilon^{\beta}\overline{\gamma}$, with $\beta > 0$, Weakly Asymmetric Simple Exclusion Process (WASEP)

For now, we consider the PASEP, the case of the WASEP with $\beta = 1/2$ is linked to the KPZ equation and studied later on.

4.3 Height function : corner growth model

Given an ASEP configuration on $\mathbb{Z}$, one can build a height function $(h(x))_{x\in\mathbb{Z}}$

$$h(0) = 0 \quad \text{and} \quad h(x+1) = \begin{cases} h(x) - 1 & \text{if } \eta_{x+1} = 1 \\ h(x) + 1 & \text{if } \eta_{x+1} = 0 \end{cases}.$$

$\mapsto$ If a particle moves from x to $x+1$ in η, the local minimum of the function h in x becomes a local maximum.

$\mapsto$ If a particle moves from $x+1$ to x in η, vice-versa.

$$h_\gamma(t,x) = h_\gamma(0,x) + 2(N_x^-(t) - N_x^+(t)),$$

where $N_x^-(t)$ is the total number of particles that came to x from $x-1$ between the times 0 and t, and $N_x^+(t)$ is the total number of particles that came to x from $x+1$ between the times 0 and t.

5 Macroscopic limit and fluctuations

Question : what is the behavior of the system at a macroscopic scale ?

First solution : given a smooth function H with bounded domain, study the behavior of

$$\varepsilon \sum_{x \in \mathbb{Z}} H(\varepsilon x)\eta_x(C_\varepsilon t) \to \int_{\mathbb{R}} H(X)\rho(T,X)dX$$

as ε goes to 0 ? $\mapsto$ Weak formulation of local equilibrium.

We denote by $X = x\varepsilon$ the macroscopic space variable and by $T = C_\varepsilon t$ the macroscopic time.

Second solution : representation by the **height function**. The macroscopic profile of the corner growth model can be written as

$$\overline{\mathbf{h}}(T,X) = \lim_{\varepsilon \to 0} \varepsilon . h_\gamma \left(\frac{T}{\gamma\varepsilon}, \frac{X}{\varepsilon} \right).$$

Hydrodynamic limit : the macroscopic profile $\overline{\mathbf{h}}$ is a weak solution to the inviscid Burgers equation

$$\partial_T \overline{\mathbf{h}} = \frac{1 - (\partial_X \overline{\mathbf{h}})^2}{2}.$$

Particular solution with wedge initial condition :

$$\overline{\mathbf{h}}(T,X) = \frac{T\left(1 + (X/T)^2\right)}{2}.$$

$\mapsto$ Faire un dessin

5.1 Back to the universality class : fluctuations

In accord with the KPZ fluctuations scale, one must consider the fluctuation around the hydrodynamic limit

$$\overline{\mathbf{f}}_0(T,Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/3} \left(h_\gamma \left(\frac{T}{\gamma\varepsilon}, \frac{Z}{\varepsilon^{2/3}} \right) - \frac{1}{\varepsilon} \overline{\mathbf{h}}(T,0) \right).$$

The compensating mean is indeed $\overline{\mathbf{h}}(T,0)$, because the scaling of the spatial fluctuation is less than the time rescaling. The fluctuation field around another macroscopic point X would be given by

$$\overline{\mathbf{f}}_X(T,Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/3} \left(h_\gamma \left(\frac{T}{\gamma\varepsilon}, \frac{X}{\varepsilon} + \frac{Z}{\varepsilon^{2/3}} \right) - \frac{1}{\varepsilon} \overline{\mathbf{h}}(T,X) \right),$$

where Z is on a mesoscopic scale relatively to X.

$\mapsto$ Completer le dessin

5.2 Long time distributions and impact of the initial conditions

Getting back to the KPZ universality class, and *long time distributions* : Another strongly presumed universality feature of the universality class, additionnaly to the scaling exponents, would be the long-time distributions of the fluctuations. The field $\bar{\mathbf{f}}_0(T, 0)$ is distributed in long time like the Tracy-Widom distribution.
$\mapsto$ A COMPLETER

5.3 Weak asymmetry and link to the KPZ equation

We now consider the case of the weak asymmetry with $\beta = \frac{1}{2}$. We now have

$$p = \frac{1}{2} + \varepsilon^{1/2}\overline{\gamma} \qquad \text{and} \qquad p = \frac{1}{2} - \varepsilon^{1/2}\overline{\gamma}.$$

Then, the fluctuation field of the weakly asymmetric exclusion process should be solution to the KPZ($\overline{\gamma}$) equation. (Bertini Giacomin '96)

More precisely this time, one considers the interface position equation

$$\overline{\mathbf{h}}^{w}(T, X) = \lim_{\varepsilon \to 0} \varepsilon . h_\gamma \left(\frac{T}{\overline{\gamma}\varepsilon^2}, \frac{X}{\varepsilon} \right).$$

Then, $\overline{\mathbf{h}}^{w}(T, X)$ evolves according to the Burgers equation

$$\partial_T \overline{\mathbf{h}}^{w} = \frac{1}{2} \Delta \overline{\mathbf{h}}^{w} + \frac{1 - (\partial_X \overline{\mathbf{h}}^{w})^2}{2}.$$

In (Bertini Giacomin '96), it is proved that for the weakly asymmetric corner growth model, the fluctuations evolve according to the KPZ equation, i.e. that letting

$$\bar{\mathbf{f}}_0(T, Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/2} \left(h_\gamma \left(\frac{T}{\overline{\gamma}\varepsilon^2}, \frac{Z}{\varepsilon} \right) - \frac{1}{\varepsilon} \overline{\mathbf{h}}^{w}(T, 0) \right),$$

the function $\bar{\mathbf{f}}_0(T, Z)$ is solution to the KPZ equation with parameter $\overline{\gamma}$.

6 Weak and Strong universality conjectures, Rescaling operator, Link with the Wilkinson Edwards universality class

$\mapsto$ Faire un dessin