Non-gradient method for hydrodynamic limits

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The purpose of these lecture notes is to present in broad terms Varadhan's non-gradient method to derive scaling limits of non gradient particle gases. Many of the

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technical proofs will not be detailed, and instead be referred to Kipnis and Landim's monograph [2, Chapter 7], where all relevant estimates are detailed.

1 Hydrodynamic for the multi-type SSEP

1.1 A non-gradient lattice gas : multi-type exclusion process

We consider here Quastel's [4] multi-type (multicolor) SSEP in two dimensions : consider two types of particles, denoted + and − for simplicity. Fix a scaling parameter N and denote by $\mathbb{T}_N^2 = \{1\dots,N\}^2$ the two-dimensional square lattice with periodic boundary conditions. The space of configurations is $\Gamma_N = \{0,1,-1\}^{\mathbb{T}_N^2}$, and we denote by $\hat{\eta} = (\hat{\eta}_x)_{x\in\mathbb{T}_N^2}$ its elements. Each site of the lattice is either occupied by a ± particle ($\hat{\eta}_x = \pm 1$), or empty ($\hat{\eta}_x = 0$). Two particles cannot coexist on the same site, regardless of their type.

We further denote by

$$\eta_x^{\pm} = \mathbf{1}_{\{\eta_x=\pm 1\}},$$

and simply by $\eta_x = \eta_x^+ + \eta_x^-$ the number of particles at site x. We consider the multi-type exclusion driven by the generator

$$\mathscr{L}_N f(\hat{\eta}) = \sum_{x\in\mathbb{T}_N^2}\sum_{|z|=1}\eta_x(1-\eta_{x+z})\left\{f\left(\hat{\eta}^{x,x+z}\right) - f(\hat{\eta})\right\},$$

where the second sum is taken over all $z = \pm e_k$, $k = 1,2$ and the configuration $\eta^{x,y}$ is obtained from η by swapping the values at x and y,

$$\hat{\eta}_{x'}^{x,y} = \begin{cases}\hat{\eta}_{x'} & \text{if } x' \neq x,y\\ \hat{\eta}_x & \text{if } x' = y\\ \hat{\eta}_y & \text{if } x' = x\end{cases}.$$

Note in particular that + and − neighboring particles *cannot swap.*

We are interested in the scaling limit of this process, started from a random configuration fitting an initial profile $\hat{\zeta} = (\zeta^+,\zeta^-)$ (see section 1.2 below).

We denote by $\mathbb{\Delta} \subset [0,1]^2$ the set of pairs $\hat{\alpha} = (\alpha^+,\alpha^-)$ of non-negative real numbers satisfying $\alpha^+ + \alpha^- \leq 1$. For any $\hat{\alpha} \in \mathbb{\Delta}$, we then denote $\alpha = \alpha^+ + \alpha^-$. The quantities $\alpha^{\pm}$ are to be thought of as respective densities of + and − particles, and α as the total particle density. Denoting by $\mathbb{T}^2 = [0,1]^2$ the continuum limit of $\mathbb{T}_N^2$, we extend the same notations to functions $\hat{\zeta} = (\hat{\zeta}^+,\hat{\zeta}^-) : \mathbb{T}^2 \to \mathbb{\Delta}$.

1.2 Initial state and definition of the process

Fix a continuous initial macroscopic profile $\hat{\zeta} = (\zeta^+,\zeta^-) : \mathbb{T}^2 \to \mathbb{\Delta}$, and define the initial distribution μ^N for the process as

$$\mu^N(\hat{\eta}) = \otimes_{x\in\mathbb{T}_N^2}\nu_{\hat{\zeta}(x/N)}(\hat{\eta}_x),$$

where for any $\hat{\alpha} \in \mathbb{A}$,

$$\nu_{\hat{\alpha}}(\hat{\eta}_x) = \begin{cases} \alpha^{\pm} & \text{if } \hat{\eta}_x = \pm 1 \\ 1-\alpha & \text{if } \hat{\eta}_x = 0 \end{cases}. \tag{1}$$

Note that we do not actually need the initial measure for the process to be a product measure, however we assume it to focus on the simplest case.

In what follows, we denote by $\hat{\eta}(t)$ a continuous-time process started from the initial distribution μ^N and driven by the accelerated generator $N^2\mathscr{L}_N$. We denote by $\mathbb{P}_{\mu^N}$ the distribution of this process, and by $\mathbb{E}_{\mu^N}$ the corresponding expectation.

We denote by $\pi^{\pm,N}$ the empirical measures of the process, $\hat{\eta}(t)$, defined for any time t as

$$\pi_t^{\pm,N} = \sum_{x\in\mathbb{T}_N^2} \eta_x^{\pm}(t)\delta_{x/N}.$$

We denote by $Q_N = \mathbb{P}_{\mu^N} \circ (\pi^{+,N}, \pi^{-,N})^{-1}$ the pushforward of the distribution $\mathbb{P}_{\mu^N}$ by the mapping $\hat{\eta} \mapsto (\pi^{+,N}, \pi^{-,N})$.

1.3 Self diffusion and diffusion coefficients

We consider an infinite volume, equilibrium SSEP at density $\rho \in [0,1]$, with a tagged particle at the origin. (two neighboring sites are exchanged at rate 1).

Definition 1 : Self-diffusion coefficient, [3]

Let us denote by $(X_1(t), X_2(t))$ the position of the tagged particle at time t. The self-diffusion coefficient is defined as the diffusion coefficient of the tagged particle, namely

$$d_s(\rho) = \lim_{t\to\infty} \frac{\mathbb{E}(X_1(t)^2)}{t}. \tag{2}$$

For any $\hat{\alpha} \in \mathbb{A}$ satisfying $\alpha > 0$, we also define

$$D^{\pm}(\hat{\alpha}) = D(\alpha^{\pm}, \alpha) := \frac{\alpha^{\pm}}{\alpha}(1 - d_s(\alpha)). \tag{3}$$

1.4 Hydrodynamic limit

To state the hydrodynamic limit, because of the lack of mixing in the system at high density, we make the following technical assumption

Assumption 1 : Bound on the initial density

We assume that the initial density profile is bounded away from 1, $\zeta(u) < 1$, i.e. $\forall u \in \mathbb{T}^2$.

Remark 1 *: Because the total density of the system is driven by the heat equation, this assumption is actually not needed in the symmetric case. We make it nonetheless for simplicity.*

Definition 2 : Weak solution to a cross-diffusive weak equation

We call a pair $\hat{\rho} = (\rho^+, \rho^-) : \mathbb{T}^2 \to \mathbb{A}$ a *weak solution to the cross-diffusive heat equation*

$$\partial_t \begin{pmatrix} \rho^+ \\ \rho \end{pmatrix} = \nabla \cdot \begin{pmatrix} d_s & D^+ \\ 0 & 1 \end{pmatrix} \nabla \begin{pmatrix} \rho^+ \\ \rho \end{pmatrix} \tag{4}$$

with initial condition $\hat{\rho}_0 = \hat{\zeta}$ if for any time t, $\rho^\pm \in H^1(\mathbb{T}^2)$, and for any smooth test functions $H : [0,T] \times \mathbb{T}^2 \to \mathbb{R}$,

$$\langle \rho_T^\pm, H_T \rangle = \langle \zeta^\pm, H_0 \rangle + \int_0^T \langle \rho_t^\pm, \partial_t H_t \rangle dt - \int_0^T \langle \nabla H_t \cdot [d_s(\rho_t) \nabla \rho_t^\pm + D^+(\hat{\rho}_t) \nabla \rho_t] \rangle dt$$

Existence and uniqueness of weak solutions to equation (4) follows for example from Amann's monograph [1].

Theorem 2 : Hydrodynamic limit for multi-type exclusion

For any smooth test function $H : \mathbb{T}^2 \to \mathbb{R}$, any $t > 0$, and any $\varepsilon > 0$, we have

$$\limsup_{N \to \infty} \mathbb{P}_{\mu^N} \left(\left| \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x^\pm(t) H(x/N) - \int_{\mathbb{T}^2} \rho_t^\pm(u) H(u) du \right| > \varepsilon \right) = 0,$$

where $\hat{\rho} = (\rho^+, \rho^-)$ is the unique weak solution to (4) with initial condition $\hat{\zeta}$, in the sense of Definition 2.

2 Non-gradient estimates

2.1 Dynkin's formula

By Dynkin's formula, one starts by writing, for any smooth test function H on $\mathbb{T}^2$,

$$\frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x^+(t) H(x/N) = \frac{1}{N^2} \sum_{x \in \mathbb{T}_N^2} \eta_x^+(0) H(x/N) + \int_0^t \sum_{x \in \mathbb{T}_N^2} H(x/N) \mathscr{L}_N \eta_x^+(s) ds + M_t^H, \tag{5}$$

where M_t^H is a martingale with quadratic variation given by

$$[M^H]_t = \frac{1}{N^2} \int_0^t ds \sum_{\substack{x \in \mathbb{T}_N^2 \\ |z|=1}} \left(H\left(\frac{x}{N}\right) - H\left(\frac{x+z}{N}\right) \right)^2 [\eta_x^+(1 - \eta_{x+z}) + \eta_{x+z}^+(1 - \eta_x)](s) ds,$$

which is of order $O(1/N^2)$, and therefore vanishes in probability. In Dynkin's formula above, the extra factor N^2 in front of the generator comes from the fact that the whole dynamics has been accelerated on a diffusive scale. We can now write

$$\mathscr{L}_N \eta_x^+ = \sum_{i=1}^{2} \left\{ j^+_{x-e_i,x} - j^+_{x,x+e_i} \right\},$$

where $j^+_{x,x+e_i}$ is the instantaneous current of $+$ particles going from x to $x+e_i$,

$$j^+_{x,x+e_i} = \eta_x^+(1-\eta_{x+e_i}) - \eta^+_{x+e_i}(1-\eta_x),$$

and therefore by (5) and the estimation of M_t^H,

$$\frac{1}{N^2}\sum_{x\in\mathbb{T}_N^2} \eta_x^+(t)H(x/N) - \frac{1}{N^2}\sum_{x\in\mathbb{T}_N^2} \eta_x^+(0)H(x/N) + \frac{1}{N}\int_0^t \sum_{x\in\mathbb{T}_N^2}\sum_{i=1}^{N} \nabla_i^N H(x) j^+_{x,x+e_i} ds,$$

vanishes in probability. Above, the discrete gradient $\nabla_i^N H$ is defined as

$$\nabla_i^N H(x) = N\left[H\left(\frac{x+e_i}{N}\right) - H\left(\frac{x}{N}\right)\right]$$

and comes from the integration by parts of the $j_{x,x+e_i}$'s. Note in particular, that since H is smooth, for any $u\in\mathbb{T}^2$, $\nabla_i^N H(\lfloor uN \rfloor)$ converges to $\partial_{u_i}H(u)$.

Because of exclusion betwee particles, the current $j^+_{x,x+e_i}$ cannot be written as a discrete gradient $\tau_{x+e_i}h - \tau_x h$ of a local function $h(\hat\eta)$. This is the main difficulty of non-gradient systems, for which the extra factor N is to be balanced out at every step. For any integer $\ell > 0$, and any configuration $\hat\eta$, define

$$\rho_x^{\pm,\ell} = \rho_x^{\pm,\ell}(\hat\eta) = \frac{1}{(2\ell+1)^2}\sum_{|y-x|\le\ell} \eta_y^{\pm} \quad \text{and} \quad \rho_x^{\ell} = \rho_x^{\ell}(\hat\eta) = \frac{1}{(2\ell+1)^2}\sum_{|y-x|\le\ell} \eta_y$$

for the density of particles $\pm$ (resp. total density) in a box of size ℓ around x. When the configuration $\hat\eta$ depends on t, we simply write $\rho_x^{\pm,\ell}(t)$ for $\rho_x^{\pm,\ell}(\hat\eta(t))$, and similarly with ρ_x^ℓ, and we denote by $\hat\rho_x^\ell$ the pair $(\rho_x^{+,\ell}, \rho_x^{-,\ell})$

2.2 The replacement Lemma

Together with a compactness argument, the only result we need to prove Theorem 2 is the following, which states that microscopic currents can be replaced in the limit by mesoscopic gradients. We will not detail in these notes the tightness estimates, in order to focus on the following replacement Lemma below.

Theorem 3 : Non-gradient replacement Lemma

Recall the definition 1 of the diffusion coefficients d_s and D, we have for any $t > 0$ and any smooth function G, we have

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N^2}G(x/N)\mathcal{W}_{x,i}^{\pm,\varepsilon N}(s)ds\right|\right)=0$$

where
$$\mathcal{W}_{x,i}^{\pm,\ell}=j_{x,x+e_i}^{\pm}+d_s(\rho_x^{\ell})\left\{\rho_{x+e_i}^{\pm,\ell}-\rho_x^{\pm,\ell}\right\}+D^{+}(\hat\rho_x^{\ell})\left\{\rho_{x+e_i}^{\ell}-\rho_x^{\ell}\right\}.$$

The proof of this result decomposes in three disctinct estimates, which we state as separate Lemmas for the sake of clarity. Given a local function ψ, we denote by s_ψ its range. We then denote, for any integer ℓ, by $\ell_\psi := \ell - s_\psi - 1$. We then denote by

$$\langle\psi\rangle_x^{\ell}=\frac{1}{\ell}\sum_{|y-x|\le\ell}\tau_y\psi,$$

its average over the box of size ℓ around x. Note in particular that $\langle\psi\rangle_x^{\ell_\psi}$ and $\langle\mathcal{L}_N\psi\rangle_x^{\ell_\psi}$ are mesurable w.r.t. $(\hat\eta_y)_{|y-x|\le\ell}$.

2.3 Non-gradient estimates

Lemma 4 : Local replacement Lemma

We have for any $t > 0$ and any smooth function G, we have

$$\inf_{\psi}\limsup_{\ell\to\infty}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N^2}G(x/N)\tau_x\widetilde{\mathcal{W}}_i^{\pm,\ell,\psi}(s)ds\right|\right)=0$$

where for any local function ψ, we denote

$$\widetilde{\mathcal{W}}_i^{\pm,\ell,\psi}=\langle j_{0,e_i}^{\pm}\rangle_0^{\ell-1}+d_s(\rho_0^{\ell})\left\{\rho_{e_i}^{\pm,\ell-1}-\rho_0^{\pm,\ell-1}\right\}+D^{\pm}(\hat\rho_0^{\ell})\left\{\rho_{e_i}^{\ell-1}-\rho_0^{\ell-1}\right\}+\langle\mathcal{L}_N\psi\rangle_0^{\ell_\psi}.$$

Lemma 5 : Non-gradient two-blocks estimate

We have for any $t > 0$ and any smooth function G, we have

$$\limsup_{\ell\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N^2}G(x/N)\tau_x\widetilde{\mathcal{V}}_i^{\pm,\ell,\varepsilon N}(s)ds\right|\right)=0,$$

where
$$\widetilde{\mathcal{V}}_i^{\pm,\ell,\varepsilon N}=D^{+}(\hat\rho_0^{\ell})\left\{\rho_{e_i}^{\ell-1}-\rho_0^{\ell-1}\right\}+D^{+}(\hat\rho_0^{\varepsilon N})\left\{\rho_{e_i}^{\varepsilon N}-\rho_0^{\varepsilon N}\right\}.$$

The same is true with d_s instead of D and with $\rho^{\pm,\ell}$ instead of ρ^ℓ.

Lemma 6 : Insertion of local averages

We have for any $t > 0$ and any smooth function G, we have

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N^2}G(x/N)\left\{j^{\pm}_{x,x+e_i}-\langle j^{\pm}_{0,e_i}\rangle_x^{\ell-1}\right\}ds\right|\right)=0.$$

and for any local function ψ,

$$\limsup_{\ell\to\infty}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t\sum_{x\in\mathbb{T}_N^2}G(x/N)\langle\mathscr{L}_N\psi\rangle_x^{\ell_\psi}ds\right|\right)=0.$$

Together, these three estimates prove Theorem 3.

We first prove Lemma 6, which is straightforward.

Proof : We simply sketch its proof. The first identity readily follows from the fact that the difference between $G(x/N)$ and its average over a box of size ℓ around x is a discrete laplacian, and is therefore of order $(\ell/N)^2$, so that the first identity is immediate. This observation also allows to remove the average around $\mathscr{L}_N\psi$. Define now the function

$$\langle G,\psi\rangle=\frac{1}{N}\sum_{x\in\mathbb{T}_N^2}G(x/N)\tau_x\psi.$$

Applying Dynkin's formula, one readily obtains that

$$M_t^{G,\psi}:=\frac{1}{N^2}\langle G,\psi\rangle(\hat\eta(t))-\frac{1}{N^2}\langle G,\psi\rangle(\hat\eta(0))-\int_0^t\mathscr{L}_N\langle G,\psi\rangle(\hat\eta(s))ds$$

is a martingale, whose quadratic variation is given by

$$[M^{G,\psi}]_t=\frac{1}{N^2}\int_0^t ds\sum_{\substack{x\in\mathbb{T}_N^2\\|z|=1}}\left(\langle G,\psi\rangle(\hat\eta^{x,x+z}(s))-\langle G,\psi\rangle(\hat\eta(s))\right)^2\left[\eta_x^+(1-\eta_{x+z})+\eta_{x+z}^+(1-\eta_x)\right](s)ds.$$

Note that the last term of the martingale is precisely the one we wish to estimate, and that the first two terms are of order $1/N$ because both G and ψ are bounded. Since ψ is a local function, for any fixed x, z, there are only a finite number of non-vanishing contributions in the gradient of $\langle G,\psi\rangle$, which is therefore of order $O(1/N)$. This yields in particular that $[M^{G,\psi}]_t$ is of order $O(1/N)$. This yields in particular that

$$\mathbb{E}_{\mu^N}\left[\left(\int_0^t\mathscr{L}_N\langle G,\psi\rangle(\hat\eta(s))ds\right)^2\right]$$

is also of order $O(1/N)$, which proves the second identity. □

2.4 Super-exponential estimates and reduction to a variational formula

The beggining of the proof of Lemmas 4 and 5 are identical, we therefore start their proof with a function $X(\hat{\eta})$ on the set of configurations, and assume that we want to estimate $\mathbb{E}_{\mu^N}\left(\left|\int_0^t X(\hat{\eta}(s))ds\right|\right)$. We start by using the entropy inequality, to obtain

$$\mathbb{E}_{\mu^N}\left(\left|\int_0^t X(\hat{\eta}(s))ds\right|\right) \leq \frac{H(\mu^N \mid \nu_\star)}{\gamma N^2} + \frac{1}{\gamma N^2} \log \mathbb{E}_{\nu_\star}\left[\exp\left(\gamma N^2 \left|\int_0^t X(\hat{\eta}(s))ds\right|\right)\right],$$

where $\nu_\star := \nu_{(1/3,1/3)}$ is an arbitrary reference measure (see (1)). Since $e^{|x|} \leq e^x + e^{-x}$, and since $\limsup \log(a_N + b_N) \leq \max\{\limsup \log a_N, \limsup \log b_N\}$ for diverging sequences a_N and b_N, the absolute values can be removed from the right-hand side in the limit. In particular, since the first term in the right-hand side is bounded by C/γ, we only need to estimate as $\gamma \to \infty$

$$\frac{1}{\gamma N^2} \log \mathbb{E}_{\nu_\star}\left[\exp\left(\gamma N^2 \int_0^t X(\hat{\eta}(s))ds\right)\right].$$

The expectation in the right-hand side, by Feynman-Kac's formula, can be written in matrix form, denoting by $\mathbf{1}$ the vector with all components equal to 1,

$$\nu_\star \exp\left(N^2 \int_0^t [\mathscr{L}_N + \gamma X](\hat{\eta}(s))ds\right)\mathbf{1} \leq \exp\left(t\gamma N^2 \lambda\right)$$

where λ designates the largest eigenvalue of the operator $X + \gamma^{-1}\mathscr{L}_N$, for which a variational formula yields

$$\mathbb{E}_{\mu^N}\left(\left|\int_0^t X(\hat{\eta}(s))ds\right|\right) \leq \frac{K}{\gamma} + t \sup_{\mathbb{E}_{\nu_\star}(\varphi^2)=1} \left\{\mathbb{E}_{\nu_\star}(\varphi^2 X) + \gamma^{-1}\mathbb{E}_{\nu_\star}(\varphi \mathscr{L}_N \varphi)\right\},$$

where we used that the initial entropy of the system is straightforwardly of the order of its volume, KN^2. The second term $\mathbb{E}_{\nu_\star}(\varphi \mathscr{L}_N \varphi)$ is (minus) the *Dirichlet form of* φ, which after a series of changes of variables $\hat{\eta} \mapsto \hat{\eta}^{x,x+z}$ can be rewritten as

$$\mathbb{E}_{\nu_\star}(\varphi \mathscr{L}_N \varphi) = -\frac{1}{2}\sum_{x,x+z} \mathbb{E}_{\nu_\star}((\nabla_{x,x+z}\varphi)^2).$$

Under this form, one easily checks that $\mathbb{E}_{\nu_\star}(\varphi \mathscr{L}_N \varphi) \leq \mathbb{E}_{\nu_\star}(|\varphi| \mathscr{L}_N |\varphi|)$, so that the supremum above can be restricted to functions of constant sign. In particular, this yields

$$\mathbb{E}_{\mu^N}\left(\left|\int_0^t X(\hat{\eta}(s))ds\right|\right) \leq \frac{K}{\gamma} + t \sup_{\varphi} \left\{\mathbb{E}_{\nu_\star}(\varphi X) + \gamma^{-1}\mathbb{E}_{\nu_\star}(\sqrt{\varphi} \mathscr{L}_N \sqrt{\varphi})\right\},$$

where this time the supremum is taken over all probability densities φ w.r.t. $\nu_\star$. Since all the required estimates above depend on a function G, which is fixed, the factor γ above can be chosen arbitrarily, and to prove that $\limsup \mathbb{E}_{\mu^N}\left(\left|\int_0^t X(s)ds\right|\right)$ vanishes, it is enough to show that

$$\limsup \sup_{\varphi} \left\{\mathbb{E}_{\nu_\star}(\varphi X) - D_N(\varphi)\right\} = 0. \tag{6}$$

where

$$D_N(\varphi) = \mathbb{E}_{\nu_\star}\big(\sqrt{\varphi}(-\mathscr{L}_N)\sqrt{\varphi}\big) = \mathbb{E}_{\nu_\star}\left(\sum_{x\in\mathbb{T}_N^2}\sum_{|z|=1}\mathbf{1}_{\{\eta_x\eta_{x+z}=0\}}\Big\{\sqrt{\varphi}(\hat\eta^{x,x+z}) - \sqrt{\varphi}(\hat\eta)\Big\}^2\right) \tag{7}$$

denotes the Dirichlet form of $\sqrt{\varphi}$.

3 Non-gradient two-blocks estimate : Lemma 5

In this section, we lay out the proof of the non-gradient two-blocks estimate stated in Lemma 5, we start from (6), with

$$X = \frac{1}{N}\sum_{x\in\mathbb{T}_N^2} G(x/N)\tau_x\widetilde{\mathcal{V}}_i^{\ell,\varepsilon N}(s),$$

and $\widetilde{\mathcal{V}}_i^{\ell,\varepsilon N}$ is the difference between local and mesoscopic gradients

$$\widetilde{\mathcal{V}}_i^{\ell,\varepsilon N} = D^+(\hat\rho_0^\ell)\Big\{\rho_{e_i}^{\ell-1} - \rho_0^{\ell-1}\Big\} + D^+(\hat\rho_0^{\varepsilon N})\Big\{\rho_{e_i}^{\varepsilon N} - \rho_0^{\varepsilon N}\Big\}.$$

First, summing by parts, we put all averages on $G(x/N)D(\rho_x^\pm,\ell)$ and $G(x/N)d_s(\rho_x^{\varepsilon N})$ to write

$$b_x = \langle G(\cdot/N)D^+(\hat\rho_\cdot^\ell)\rangle_x^{\ell-1} - \langle G(\cdot/N)D^+(\hat\rho_\cdot^{\varepsilon N})\rangle_x^{\varepsilon N}.$$

To prove Lemma 5, it suffices to show that

$$\limsup_{\ell\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_{\varphi}\left\{\frac{1}{N}\mathbb{E}_{\nu_\star}\left(\varphi\sum_{x\in\mathbb{T}_N^2} b_x(\eta_{x+e_i}-\eta_x)\right) - D_N(\varphi)\right\} = 0.$$

In order to make the jump rates appear, rewrite $\eta_{x+e_i}-\eta_x = \eta_{x+e_i}(1-\eta_x)-\eta_x(1-\eta_{x+e_i})$, and by a change of variable $\hat\eta\mapsto\hat\eta^{x,x+e_i}$, the first expectation can be written as

$$\begin{aligned}
&\frac{1}{N}\sum_{x\in\mathbb{T}_N^2}\mathbb{E}_{\nu_\star}\left(\eta_x(1-\eta_{x+e_i})\left[\varphi b_x(\hat\eta^{x,x+e_i}) - \varphi b_x\right]\right)\\
&\le \frac{1}{N}\sum_{x\in\mathbb{T}_N^2}\mathbb{E}_{\nu_\star}\left(\varphi(\hat\eta^{x,x+e_i})|b_x(\hat\eta^{x,x+e_i}) - b_x|\right) + \mathbb{E}_{\nu_\star}\left(\eta_x(1-\eta_{x+e_i})b_x\left[\varphi(\hat\eta^{x,x+e_i}) - \varphi(\hat\eta)\right]\right).
\end{aligned} \tag{8}$$

Note that because we slightly reduced the range of the average in ℓ, $\hat\rho_y^\ell$ it is unchanged by jumps between x and $x+e_i$ if $|y-x|\le\ell-1$. In particular, Since the diffusion coefficient is smooth, we have the rough estimate

$$b_x(\hat\eta^{x,x+e_i}) - b_x = \frac{1}{(1+\varepsilon N)^2}\sum_{|y-x|\le\varepsilon N} H(y/N)\left[D^+(\hat\rho_y^{\varepsilon N}\pm 1/(1+\varepsilon N)^2) - D^+(\hat\rho_y^{\varepsilon N})\right],$$

which is of order $1/(\varepsilon N)^2$ because H is bounded. In particular, the corresponding contribution in (8) vanishes, and we are left with estimating the second one. The second one is rather straightforward : we use the elementary inequality

$$BC \le \frac{A}{2}B^2 + \frac{1}{2A}C^2,$$

which holds for any $A > 0$, B, C, and obtain

$$\eta_x(1-\eta_{x+e_i})b_x\left[\varphi(\hat{\eta}^{x,x+e_i})-\varphi(\hat{\eta})\right] \leq \frac{b_x^2}{2N}[\varphi(\hat{\eta}^{x,x+e_i})+\varphi]^2+\frac{N}{2}\eta_x(1-\eta_{x+e_i})[\sqrt{\varphi}(\hat{\eta}^{x,x+e_i})-\sqrt{\varphi}]^2.$$

The contribution of the second term in (8) is

$$\frac{1}{2}\mathbb{E}_{\nu_\star}\left(\sum_{x\in\mathbb{T}_N^2}\eta_x(1-\eta_{x+e_i})\left\{\sqrt{\varphi}(\hat{\eta}^{x,x+z})-\sqrt{\varphi}(\hat{\eta})\right\}^2\right)\leq\frac{1}{2}D_N(\varphi),$$

whereas the contribution of the first term first term, now that we have gained the extra factor N that we were missing, is less than

$$\frac{1}{N^2}\sum_{x\in\mathbb{T}_N^2}\mathbb{E}_{\nu_\star}\left((b_x^2(\hat{\eta}^{x,x+e_i})+b_x^2)\varphi\right)\simeq\frac{2}{N^2}\sum_{x\in\mathbb{T}_N^2}\mathbb{E}_{\nu_\star}\left(b_x^2\varphi\right).$$

Now that we have balanced out the extra factor N, we only need to estimate

$$\frac{2}{N^2}\sum_{x\in\mathbb{T}_N^2}\mathbb{E}_{\nu_\star}\left(\varphi b_x^2\right)-\frac{1}{2}D_N(\varphi).$$

It is straightforward to see that this quantity is non positive if $D_N(\varphi)$ is not bounded. If, instead, $D_N(\varphi)$ is bounded, the usual two-blocks estimate allows to conclude.

Note that we cheated a little bit because we chose the "easy" gradient, in the sense that $\eta_{x+e_i}-\eta_x=\eta_{x+e_i}(1-\eta_x)-\eta_x(1-\eta_{x+e_i})$ makes the jump rates automatically appear, and we did not have to use an integration by parts formula. This is not the case for the other gradient, for which one needs to introduce empty sites in the configuration. This is not a problem in the symmtric case, however, since the heat equation satisfies the maximum principle, and regions of high density are not frequent. In order not to burden these notes, we leave this significant hurdle aside.

4 Local replacement Lemma

4.1 Integration by parts formula

In what follows, we define

$$C_0=\{\psi,\mathbb{E}_{\hat{K},\ell}(\psi)=0\ \forall\hat{K},\ \forall\ell>s_\psi\}$$

the space of local functions with mean 0 w.r.t. all canonical measures $\mathbb{E}_{\hat{K},\ell}$. Note in particular that $\eta_{e_i}^+-\eta_0^+$ and the current j_{0,e_i} are in C_0, as well as any $\mathscr{L}\psi$ for a local function ψ. An important property of any function ψ in C_0 is that it is in the range of the generator $\mathscr{L}_{s_\psi}$ restricted to its domain. To see that, fix a hyperplane $\Sigma_{\hat{K},s_\psi}$ with $\hat{K}$ particles in B_{s_ψ}. The kernel of $\mathscr{L}_\ell$ is composed of constant functions over $\Sigma_{\hat{K},\ell}$, because by ergodicity $\mathscr{L}_\ell\psi=0$ means in particular that the dirichlet form of ψ over B_ℓ vanishes, so that ψ is unchanged by moving particles in the system. Note that this is no longer the case when there is no empty sites in B_ℓ, because the configuration can no longer mix. However, in order not to burden these notes, we do not tackle in detail this issue, the main idea is that full clusters are extremely unlikely under a product measure with density less than 1. For any fixed $\hat{K}$, the range of $\mathscr{L}_\ell$ therefore has codimension 1 in $\Sigma_{\hat{K},\ell}$, and since it is included in the set of mean-0 function, those two must be identical. Denote $\nabla_{x,y} f = \eta_x(1-\eta_y)\left(f(\hat{\eta}^{x,y}) - f(\hat{\eta})\right)$.

For any function ψ in C_0, since ψ is in the range of $\mathscr{L}_{s_\psi}$, we have the following integration by parts formula, which holds for any function h and any ℓ larger than both s_ψ and s_h :

$$\mathbb{E}_{\hat{K},\ell}(h\psi) = \sum_{x,x+z\in B_{s_\psi}} \mathbb{E}_{\hat{K},\ell}\left((\nabla_{x,x+z}[-\mathscr{L}_{s_\psi}^{-1}\psi])\nabla_{x,x+z}h\right) \tag{9}$$

Further note that the "integral" $-\nabla_{x,x+z}\mathscr{L}_{s_\psi}^{-1}\psi$ of ψ is also a local function, so that its L^2 norm is bounded by a constant depending on ψ.

4.2 Projection on local hyperplanes

We now turn to the main difficulty of the non-gradient method, namely the local replacement Lemma 4, which allows to replace the at the local scale the instantaneous current by a gradient quantity. We will center the proof around the currents of + particles, naturally, the current of − particles is treated in the same way. Recall that we want to prove that

$$\inf_\psi \limsup_{\ell\to\infty} \limsup_{N\to\infty} \mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\int_0^t \sum_{x\in\mathbb{T}_N^2} G(x/N)\tau_x \widetilde{\mathcal{W}}_i^{\ell,\psi}(s)ds\right|\right) = 0$$

where

$$\widetilde{\mathcal{W}}_i^{\ell,\psi} = \langle j^+_{0,e_i}\rangle_0^{\ell-1} + d_s(\rho_0^\ell)\left\{\rho_{e_i}^{\ell-1} - \rho_0^{\pm,\ell-1}\right\} + D^+(\hat{\rho}_0^\ell)\left\{\rho_{e_i}^{\ell-1} - \rho_0^{\ell-1}\right\} + \langle \mathscr{L}_N\psi\rangle_0^{\ell_\psi}.$$

Once again, our starting point is (6), thanks to which it is enough to show that

$$\inf_\psi \limsup_{\ell\to\infty} \limsup_{N\to\infty} \sup_\varphi \left\{\mathbb{E}_{\nu_\star}(\varphi X^\psi) - D_N(\varphi)\right\} = 0,$$

with

$$X^\psi(\hat{\eta}) = \frac{1}{N}\sum_{x\in\mathbb{T}_N^2} G(x/N)\tau_x\widetilde{\mathcal{W}}_i^{\ell,\psi}.$$

Now shorten φ^x for $\tau_{-x}\varphi$, we can write

$$\mathbb{E}_{\nu_\star}(\varphi X^\psi) = \frac{1}{N}\sum_{x\in\mathbb{T}_N^2} G(x/N)\mathbb{E}_{\nu_\star}\left(\widetilde{\mathcal{W}}_i^{\ell,\psi}\varphi^x\right).$$

Note that $\widetilde{\mathcal{W}}_i^{\ell,\psi}$ only depends on the configuration through sites in $B_\ell = \{-\ell,\dots,\ell\}^2$, we now need to project on hyperplanes with fixed number of particles. For any pair of integers $\hat{K} = (K^+, K^-)$ satisfying $\hat{K}/|B_\ell| \in \mathbb{A}$, we condition φ_x to hyperplanes $\Sigma_{\hat{K},\ell}$ with $\hat{K}$ particles, namely for any configuration $\sigma\Sigma_{\hat{K},\ell}$ on B_ℓ with $\hat{K}$ particles,

$$\varphi^x_{\hat{K},\ell}(\sigma) = \frac{\mathbb{E}_{\nu_\star}(\varphi^x \mathbf{1}_{\{\hat{\eta}_{|B_\ell} = \sigma\}})}{\mathbb{E}_{\nu_\star}(\varphi^x \mathbf{1}_{\{\hat{\rho}_\ell = \hat{K}/|B_\ell|\}})}.$$

We further denote by

$$m^x_{\hat{K},\ell} = \mathbb{E}_{\nu_\star}(\varphi^x \mathbf{1}_{\{\hat{\rho}_\ell = \hat{K}/|B_\ell|\}}),$$

the probability to have $\hat{K}$ particles in B_ℓ under $\varphi^x d\nu_\star$. With these notations, we can rewrite

$$\mathbb{E}_{\nu_\star}(\varphi X^\psi) = \frac{1}{N} \sum_{x \in \mathbb{T}^2_N} \sum_{\hat{K}} G(x/N) m^x_{\hat{K},\ell} \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}^{\ell,\psi}_i \varphi^x_{\hat{K},\ell}\right),$$

and by convexity of the Dirichlet form, we also have the bound

$$\sum_{x \in \mathbb{T}^2_N} \sum_{\hat{K}} m^x_{\hat{K},\ell} D_{\hat{K},\ell}(\varphi^x_{\hat{K},\ell}) \leq |B_\ell| D_N(\varphi),$$

where $D_{\hat{K},\ell}$ is the Dirichlet form on B_ℓ w.r.t. the canonical measure,

$$D_{\hat{K},\ell}(f) = \mathbb{E}_{\hat{K},\ell}\left(\sum_{x, x+z \in B_\ell} \mathbf{1}_{\{\eta_x \eta_{x+z} = 0\}} \left[\sqrt{f}(\hat{\eta}^{x,x+z}) - \sqrt{f}\right]^2\right)$$

with

$$\nu_{\hat{K},\ell} := \nu_\star\Big(\hat{\eta}_{|B_\ell}) = \cdot \;\Big|\; \hat{\rho}_\ell = \hat{K}/|B_\ell|\Big),$$

and $\mathbb{E}_{\hat{K},\ell}$ designates the corresponding expectation.

In particular,

$$\begin{aligned}\mathbb{E}_{\nu_\star}(\varphi X^\psi) - D_N(\varphi) &\leq \frac{1}{|B_\ell|} \sum_{x \in \mathbb{T}^2_N} \sum_{\hat{K}} m^x_{\hat{K},\ell} \left[\beta_x \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}^{\ell,\psi}_i \varphi^x_{\hat{K},\ell}\right) - D_{\hat{K},\ell}(\varphi^x_{\hat{K},\ell})\right] \\ &\leq \frac{1}{|B_\ell|} \sum_{x \in \mathbb{T}^2_N} \sup_{\hat{K}} \sup_h \mathbb{E}_{\hat{K},\ell}\left(\sqrt{h}\left[\beta_x \widetilde{\mathcal{W}}^{\ell,\psi}_i + \mathscr{L}_\ell\right] \sqrt{h}\right)\end{aligned}$$

where $\beta_x = G(x/N)|B_\ell|/N$, and the supremum is taken over all densities h w.r.t. $\nu_{\hat{K},\ell}$. Note that the supremum over h is less than the largest eigenvalue of the operator $\beta_x \widetilde{\mathcal{W}}^{\ell,\psi}_i + \mathscr{L}_\ell$. In order not to overburden with technical details, we will skip some details of the following arguments, and refer the reader to [2] for a detailed implementation. Since β_x is small and $\mathscr{L}_\ell$'s largest eigenvalue is 0, and admits constant functions as eigenvectors, we expect that the unitary eigenvector $\sqrt{h_x}$ for the largest eigenvalue $\beta_x \widetilde{\mathcal{W}}^{\ell,\psi}_i + \mathscr{L}_\ell$ can be approximately written as $\sqrt{h_x} \equiv 1 + g_x$. In particular, we must then have that

$$\sup_h \mathbb{E}_{\hat{K},\ell}\left(\sqrt{h}\left[\beta_x \widetilde{\mathcal{W}}^{\ell,\psi}_i + \mathscr{L}_\ell\right] \sqrt{h}\right) \leq \beta_x \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}^{\ell,\psi}_i h_x\right) - D_{\hat{K},\ell}(h_x).$$

Since $\widetilde{\mathcal{W}}_i^{\ell,\psi} \in C_0$ it is in the range of $\mathscr{L}_\ell$ (see below), so that we can write, at least formally,

$$
\begin{aligned}
\mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}_i^{\ell,\psi} h_x\right) &= \mathbb{E}_{\hat{K},\ell}\left([\mathscr{L}_\ell^{-1}\beta_x \widetilde{\mathcal{W}}_i^{\ell,\psi}]\mathscr{L}_\ell h_x\right) \\
&= \sum_{x,x+z\in B_\ell} \mathbb{E}_{\hat{K},\ell}\left(\nabla_{x,x+z}[-\mathscr{L}_\ell^{-1}\beta_x \widetilde{\mathcal{W}}_i^{\ell,\psi}]\nabla_{x,x+z}h_x\right) \\
&\leq \sum_{x,x+z\in B_\ell} \mathbb{E}_{\hat{K},\ell}\left((\nabla_{x,x+z}[-\mathscr{L}_\ell^{-1}\beta_x \widetilde{\mathcal{W}}_i^{\ell,\psi}])^2\right) + \frac{1}{4}\mathbb{E}_{\hat{K},\ell}\left((\nabla_{x,x+z}h_x)^2\right) \\
&\leq \beta_x^2 \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}_i^{\ell,\psi}(-\mathscr{L}_\ell^{-1})\widetilde{\mathcal{W}}_i^{\ell,\psi}\right) + \frac{1}{4}\mathbb{E}_{\hat{K},\ell}\left(h_x(-\mathscr{L}_\ell)h_x\right)
\end{aligned}
$$

Assuming that $\sqrt{h_x} \equiv 1 + g_x$ for a small perturbation g_x, to leading order, we have $\mathbb{E}_{\hat{K},\ell}\left(h_x(-\mathscr{L}_\ell)h_x\right) \equiv 4D_{\hat{K},\ell}(h_x)$, so that finally we obtain in the limit $N \to \infty$

$$
\sup_h \mathbb{E}_{\hat{K},\ell}\left(\sqrt{h}\left[\beta_x \widetilde{\mathcal{W}}_i^{\ell,\psi} + \mathscr{L}_\ell\right]\sqrt{h}\right) \leq \beta_x^2 \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}_i^{\ell,\psi}(-\mathscr{L}_\ell^{-1})\widetilde{\mathcal{W}}_i^{\ell,\psi}\right),
$$

and therefore by definition of β_x

$$
\limsup_{N\to\infty} \sup_\varphi \left\{\mathbb{E}_{\nu_\star}(\varphi X^\psi) - D_N(\varphi)\right\} \leq \|G\|_\infty^2 \sup_{\hat{K}} |B_\ell| \mathbb{E}_{\hat{K},\ell}\left(\widetilde{\mathcal{W}}_i^{\ell,\psi}(-\mathscr{L}_\ell^{-1})\widetilde{\mathcal{W}}_i^{\ell,\psi}\right). \tag{10}
$$

The main objective of the non-gradient method is to prove that the right-hand side above vanishes as $\ell \to \infty$ provided one optimizes over all functions ψ, which proves the two-blocks estimate stated in Lemma 5.

4.3 Estimation of the local variance

Fix $\hat{\alpha} \in \mathbb{\Delta}$, and a sequence $(\hat{K}_\ell)$ such that $\hat{K}_\ell/|B_\ell| \to \hat{\alpha}$. We claim that given a local function $\psi \in C_0$,

$$
\ll \psi \gg_{\hat{\alpha}} := \limsup_{\ell\to\infty} \frac{1}{|B_\ell|}\mathbb{E}_{\hat{K}_\ell,\ell}\left(\sum_{x\in B_{\ell_\psi}} \tau_x\psi, (-\mathscr{L}_\ell^{-1}) \sum_{x\in B_{\ell_\psi}} \tau_x\psi\right) \tag{11}
$$

does not depend on the chosen sequence $(\hat{K}_\ell)$. Furthermore, $\ll \psi \gg_{\hat{\alpha}}^{1/2}$ is a semi-norm on C_0. We will admit those two statements, in order to focus on the structure of the space C_0 quotiented by the kernel of $\ll \cdot \gg_{\hat{\alpha}}$. Provided the two previous statements are true, in order to prove that the right-hand side of (10) vanishes, it is enough to show that

$$
\inf_\psi \sup_{\hat{\alpha}\in\mathbb{\Delta}} \ll j_{0,e_i}^+ + d_s(\alpha)(\eta_{e_i}^+ - \eta_0^+) + D^\pm(\hat{\alpha})(\eta_{e_i} - \eta_0) + \mathscr{L}\psi \gg_{\hat{\alpha}} = 0 \tag{12}
$$

Define $\mathcal{H}_{\hat{\alpha}} = C_0/Ker \ll \cdot \gg_{\hat{\alpha}}$. We claim that

$$
\mathcal{H}_{\hat{\alpha}} = \overline{J \oplus \mathscr{L}C}, \tag{13}
$$

where C is the set of local functions and J is the range of particle currents,

$$
J = \left\{\sum_i a_i j_{0,e_i} + b_i j_{0,e_i}^+,\ a, b \in \mathbb{R}^2\right\}.
$$

In this identity, $j_{0,e_i} = j^+_{0,e_i} + j^-_{0,e_i} = \eta_{e_i} - \eta_0$ is the total particle current.

In order not to burden the proof, we will drop many technical steps, to focus on the main ideas. Recall that any $\psi \in \mathcal{C}_0$ is in the range of the generator $\mathscr{L}_\psi$, and rewrite the expectation in (11)

$$\frac{1}{|B_\ell|}\sum_{y,y+e_i\in B_\ell}\mathbb{E}_{\hat{K},\ell}\left(\left[\nabla_{y,y+e_i}(-\mathscr{L}_\ell^{-1})\sum_{x\in B_{\ell_\psi}}\tau_x\psi\right]^2\right).$$

Imagine for a moment that $\mathscr{L}_\ell^{-1}$ is a local operator, and can therefore be replaced by its infinite volume counterpart in the expecatation above. Then, far away from the boundary of B_ℓ, and assuming that $\hat{K}_\ell/|B_\ell| \to \hat{\alpha}$, by translation invariance of the limiting measure $\nu_{\hat{\alpha}}$, each contribution for fixed $y, y+e_i$ in the first sum above would converge to

$$\mathbb{E}_{\nu_{\hat{\alpha}}}\left(\left[\nabla_{0,e_i}(-\mathscr{L}^{-1})\sum_{x\in\mathbb{Z}^2}\tau_x\psi\right]^2\right).$$

In particular, letting $\Sigma_\psi = \sum_{x\in\mathbb{Z}^2}\tau_x\psi$, which is not well defined, but whose gradient or integral against a local function is, one would formally obtain

$$\ll \psi \gg_{\hat{\alpha}} = \mathbb{E}_{\nu_{\hat{\alpha}}}\left(\left[\nabla_{0,e_1}(-\mathscr{L}^{-1})\Sigma_\psi\right]^2 + \left[\nabla_{0,e_2}(-\mathscr{L}^{-1})\Sigma_\psi\right]^2\right) = \|\mathscr{F}(\psi)\|_{2,\hat{\alpha}},$$

where $\mathfrak{F}(\psi)$ denotes the formal vector

$$\mathfrak{F}(\psi) = -(\nabla_{0,e_1}\mathscr{L}^{-1}\Sigma_\psi, \nabla_{0,e_2}\mathscr{L}^{-1}\Sigma_\psi),$$

and $\|\cdot\|_{2,\hat{\alpha}}$ is its $L^2(\nu_{\hat{\alpha}}\otimes\nu_{\hat{\alpha}})$ norm. Of course, this reasoning fails because $\mathscr{L}^{-1}$ is not a local operator, and to be made sense of, the formal computations above need to be drawn out through variational formulas. However, in some specific cases where $\mathscr{L}^{-1}\Sigma_\psi$ is explicit, the formal computations above can me made rigorous.

4.4 Norm of the currents and $\mathscr{L}\psi$

We now consider such cases. Associate with $\ll \psi \gg_{\hat{\alpha}}$ the corresponding inner product on $\mathcal{C}_0$

$$\ll \psi, \phi \gg_{\hat{\alpha}} := \limsup_{\ell\to\infty}\frac{1}{|B_\ell|}\mathbb{E}_{\hat{K}_\ell,\ell}\left(\sum_{x\in B_{\ell_\psi}}\tau_x\psi, (-\mathscr{L}_\ell^{-1})\sum_{x\in B_{\ell_\phi}}\tau_x\phi\right), \tag{14}$$

so that $\ll \psi \gg_{\hat{\alpha}} = \ll \psi, \psi \gg_{\hat{\alpha}}$. Now, let us consider first a local function $\phi \in \mathcal{C}$, then $\mathscr{L}\phi$ is also a local function and furthermore belongs to $\mathcal{C}_0$. In particular, for any $x \in B_{\ell_\phi}$ we have $\mathscr{L}\tau_x\phi = \mathscr{L}_\ell\tau_x\phi$, so that

$$\ll \psi, \mathscr{L}\phi \gg_{\hat{\alpha}} := \limsup_{\ell\to\infty}\frac{-1}{|B_\ell|}\mathbb{E}_{\hat{K}_\ell,\ell}\left(\sum_{x\in B_{\ell_\psi}}\tau_x\psi, \sum_{x\in B_{\ell_\phi}}\tau_x\phi\right) = -\mathbb{E}_{\hat{\alpha}}(\psi\Sigma_\phi), \tag{15}$$

where once again $\Sigma_\psi = \sum_{x\in\mathbb{Z}^2} \tau_x\psi$, is a formal sum whose expectation against any mean-0 local function (like ψ) is well defined. This allows us to define the inner product $\ll \cdot, \mathscr{L}\phi \gg_{\hat{\alpha}}$ for any local function $\phi \in C$, and the formal function $\mathfrak{F}(\mathscr{L}\phi)$ defined in the previous section as

$$\mathfrak{F}(\mathscr{L}\phi) = -(\nabla_{0,e_1}\Sigma_\phi, \nabla_{0,e_2}\Sigma_\phi),$$

which is a well defined function, because ϕ is a local function. In particular, the previous identity yields that for any local function ψ,

$$\ll \eta^\pm_{e_i} - \eta^\pm_0, \mathscr{L}\phi \gg_{\hat{\alpha}} = 0,$$

so that in $\mathcal{H}_{\hat{\alpha}}$, the gradients are orthogonal to $\mathscr{L}C$, where C is the set of local functions.

We now turn to the currents. We will focus on j^+_{0,e_i}, naturally the other currents can be treated in the same way. The crucial identity

$$\mathscr{L}_\ell \sum_{x\in B_\ell} x_i\eta^+_x = \sum_{x,x+e_i\in B_\ell} j^+_{x,x+e_i}$$

yields, up to controllable boundary terms, that

$$\ll \psi, j_{0,e_i} \gg_{\hat{\alpha}} := \limsup_{\ell\to\infty} \frac{-1}{|B_\ell|}\mathbb{E}_{\hat{K}_\ell,\ell}\left(\sum_{x\in B_{\ell_\psi}} \tau_x\psi, \sum_{x\in B_{\ell_\phi}} x_i\eta^+_x\right) = -\mathbb{E}_{\hat{\alpha}}\left(\psi\sum_{x\in\mathbb{Z}^2} x_i\eta^+_x\right),$$

which once again is well defined for any mean-0 local function ψ. This yields that, given once again the formal function $\mathfrak{F}$ defined in the previous section, we have the identity

$$\mathfrak{F}(j_{0,e_1}) = -\left(\nabla_{0,e_1}\sum_{x\in\mathbb{Z}^2} x_1\eta^+_x, \nabla_{0,e_2}\sum_{x\in\mathbb{Z}^2} x_1\eta^+_x\right) = -(\eta^+_0(1-\eta_{e_1}), 0),$$

$$\mathfrak{F}(j_{0,e_2}) = -\left(\nabla_{0,e_1}\sum_{x\in\mathbb{Z}^2} x_2\eta^+_x, \nabla_{0,e_2}\sum_{x\in\mathbb{Z}^2} x_2\eta^+_x\right) = -(0, \eta^+_0(1-\eta_{e_2})),$$

5 Closed forms on the space of configurations

5.1 Discrete differential forms

We now need to formalize the definition of $\mathscr{F}$. To do so, we introduce the concept of *translation invariant closed differential form* in the context of particle systems. Define the set $\Sigma_\infty = \{\hat{\eta}_x, z \in \mathbb{Z}^2\}$ the set of infinite volume confugurations for our model. We consider the graph $G = (\Sigma_\infty, E)$ with set of edges given by $(\hat{\eta}, \hat{\eta}') \in E$ iff there exists $x, x+z \in \mathbb{Z}$ such that $\hat{\eta}' = \hat{\eta}^{x,x+z}$ and $\eta_x(1-\eta_{x+z}) = 1$.

Throughout this section, we fix a grand-canonical parameter $\hat{\alpha}$. By abuse of notation, we still denote by $\nu_{\hat{\alpha}}$ the grand-canonical measure on $\mathbb{Z}^2$. Let $(u_{x,x+z})_{x\in\mathbb{Z}^2,|z|=1}$

be a collection of random variables in $L^2(\nu_{\hat\alpha})$, we assume that $u_{x,x+z}(\hat\eta)$ vanishes whenever $\eta_x(1-\eta_{x+z})=0$. Thusly defined, $u_{x,x+z}$ can be seen as a function on the family of edges $(\hat\eta, \hat\eta^{x,x+z}) \in E$.

Given a configuration $\hat\eta := \hat\eta^(0)$, and a sequence γ of licit successive particle jumps $x_i \mapsto x_i + z_i$ yielding successive configurations $\hat\eta^(1) \dots, \hat\eta^{(n)}$, we define the integral of u over the path γ as

$$I_{\gamma,u}(\hat\eta) = \sum_{i=1}^{n} u_{x_i, x_i+z_i}(\hat\eta^{(i-1)}),$$

which can be thought of as the "cost" for u to perform the sequence of jumps γ. We call u a *closed differential form* if for any closed path γ (i.e. such that $\hat\eta^{(n)} = \hat\eta^{(0)}$) of licit jumps, we have

$$I_{\gamma,u}(\hat\eta) = 0 \quad \nu_{\hat\alpha} - a.s.$$

A natural example of a closed form is an *exact form* : fix a local function ψ, one can define

$$u^{\psi}_{x,x+z} = \nabla_{x,x+z}\psi = \eta_x(1-\eta_{x+z})\left[\psi(\hat\eta^{x,x+z}) - \psi(\hat\eta)\right],$$

whose integral over a path $\gamma = \hat\eta^{(0)}, \dots, \hat\eta^{(n)}$ of licit jumps equals $f(\hat\eta^{(n)}) - f(\hat\eta^{(n)})$, and in particular vanishes if the path is closed.

We will now be interested in a special class of translation invariant closed forms, called *germs of closed forms*;

Definition 3 : Germs of closed and exact forms

A pair (u_1, u_2) of functions in $L^2(\nu_{\hat\alpha})$ is called the germ of a closed form if the differential form $u_{x,x+z}$ defined by

$$u_{x,x+e_i} = \tau_x u_i \quad \text{ and } u_{x+e_i,x}(\hat\eta) = -\tau_x u_i(\hat\eta^{x,x+e_i}) = -u_{x,x+e_i}(\hat\eta^{x,x+e_i})$$

is a closed form. We endow the set of germs of closed form $\mathfrak{C}_{\hat\alpha}$ with its L^2 norm

$$||u||^2 = \mathbb{E}_{\nu_{\hat\alpha}}(u_1^2 + u_2^2).$$

A particular exemple of a germ of a closed form is the germ of an exact form : consider ψ a local function, and recall that we defined the formal sum $\Sigma_\psi = \sum_{x\in\mathbb{Z}^2} \tau_x\psi$, whose gradient $\nabla_{x,x+z}$ is well defined for any $x, x+z$. Then, the function

$$(u_1^\psi, u_2^\psi) = \nabla\Sigma_\psi := (\nabla_{0,e_1}\Sigma_\psi, \nabla_{0,e_2}\Sigma_\psi)$$

is a germ of a closed form. We denote by $\mathfrak{E}_{\hat\alpha}$ the set of germs of exact forms.

The main result of this section is that, once restricted to a set of functions on which the spectral gap holds, we can write

$$\mathfrak{C}_{\hat\alpha} = \overline{\mathfrak{E}_{\hat\alpha}} \oplus \mathfrak{J}, \tag{16}$$

where

$$\mathfrak{J} = \{a_1\mathfrak{j}_1 + a_2\mathfrak{j}_2 + b_1\mathfrak{j}_1^+ + b_2\mathfrak{j}_2^+, \quad \text{for } \; a_1, a_2, b_1, b_2 \in \mathbb{R}\}$$

is the linear span of the four germs of closed forms

$$\mathfrak{j}_1 = (\eta_0(1-\eta_1), 0) \qquad \mathfrak{j}_2 = (0, \eta_0(1-\eta_2))$$

$$\mathfrak{j}_1^+ = (\eta_0^+(1-\eta_1), 0) \qquad \mathfrak{j}_2^+ = (0, \eta_0^+(1-\eta_2)),$$

and $\overline{\mathfrak{E}_{\hat{\alpha}}}$ denotes the closure in $L^2(\nu_{\hat{\alpha}})$ of $\mathfrak{E}_{\hat{\alpha}}$. Note that these from forms represent the transportation of (+ and total) mas throughout the system : once expanded into a closed form, the corresponding close form give weight 1 (resp. -1) to any edge representing a jump in the forward (resp. backward) direction.

More precisely, the four germs of closed forms $\mathfrak{j}_i$, $\mathfrak{j}_i^+$ generate the closed forms $\nabla_{x,x+z}f_i$, $\nabla_{x,x+z}f_i^+$ defined by the formal functions

$$f_i = \sum_{x\in\mathbb{Z}^2} x_i\eta_i \qquad \text{and} \qquad f_i^+ = \sum_{x\in\mathbb{Z}^2} x_i\eta_i^+.$$

We start by considering the case of the finite volume graph.

Lemma 7 : Exactness of closed forms on a finite box

Consider the graph $G_n = (\Sigma_n, E_n)$ in finite volume with closed boundary conditions. Any closed form on G_n is an exact form.

This is fairly straightforward to prove, by choosing a closed form u and explicitely building a function f_u on all connected hyperplanes with fixed number of particles : for any $\hat{K}$ on B_n, we choose an arbitrary configuration $\hat{\eta}_{\hat{K}}$ for which we set $f(\hat{\eta}_{\hat{K}}) = 0$. We then define $f(\hat{\eta})$ as the path integral from $\hat{\eta}_{\hat{K}}$ to $\hat{\eta}$, which does not depend on the chosen path because u is closed.

5.2 Proof of Equation (12).

Before sketching the proof of the decomposition 16, we show that the decomposition (13) is its consequence. Recall that ψ is a function in C_0, we want to study give meaning to

$$\mathfrak{F}(\psi) = -(\nabla_{0,e_1}\mathscr{L}^{-1}\Sigma_\psi, \nabla_{0,e_2}\mathscr{L}^{-1}\Sigma_\psi). \tag{17}$$

This quantity is a priori not well defined because $\mathscr{L}^{-1}$ is not a local operator, but is characterized through variational formulas as the limit point in $L^2(\nu_{\hat{\alpha}})$ of well-defined functions, which we can formally write as

$$\mathfrak{F}_\ell(\psi) := -(\nabla_{0,e_1}\mathscr{L}_\ell^{-1}\Sigma_{\ell,\psi}, \nabla_{0,e_2}\mathscr{L}_\ell^{-1}\Sigma_{\ell,\psi}),$$

where $\Sigma_{\ell,\psi} = \sum_{x\in B_{\ell_\psi}} \tau_x\psi$. Of course, the limiting quantity $\mathfrak{F}(\psi)$ is difficult to characterize and cannot be defined by (17), however thanks to the gradient in $\mathfrak{F}_\ell(\psi)$, it is not hard to show that the limit point must be the germ of a closed form in the sense of Definition 3, and can therefore be decomposed according to (16).

In particular, since $\eta_{e_i} - \eta_0 \in C_0$, there must exist a sequence of local functions $(\psi_k^{i,+})_{k\in\mathbb{N}}$, and four coefficients $a_1^{i,+}$, $a_2^{i,+}$, $b_1^{i,+}$, $b_2^{i,+}$, such that in $L^2(\nu_{\hat{\alpha}})$,

$$\mathfrak{F}(\eta_{e_i}^+ - \eta_0^+) = a_1^{i,+}\mathfrak{j}_1 + a_2^{i,+}\mathfrak{j}_2 + b_1^{i,+}\mathfrak{j}_1^+ + b_2^{i,+}\mathfrak{j}_1^+ + \lim_{k\to\infty}\nabla\Sigma_{\psi_k^{i,+}}$$

As we have already seen, $\nabla\Sigma_\psi = \mathfrak{F}(\mathscr{L}\psi)$, and

$$\mathfrak{j}_i = \mathfrak{F}(j_{0,e_i}) \quad \text{and} \quad \mathfrak{j}_i^+ = \mathfrak{F}(j_{0,e_i}^+),$$

so that one obtains

$$\limsup_{k\to\infty} \ll \eta_{e_i}^{\pm} - \eta_0^{\pm} - (a_1^{i,\pm} j_{0,e_1} + a_2^{i,\pm} j_{0,e_2} + b_1^{i,\pm} j_{0,e_1}^+ + b_2^{i,\pm} j_{0,e_2}^+) - \mathscr{L}\psi_k^{i,\pm} \gg_{\hat{\alpha}} = 0,$$

and we further have

$$\ll \eta_{e_i} - \eta_0 + j_{0,e_i} \gg_{\hat{\alpha}} .$$

Since all the inner products are known (see Subsection 4.4), one can (after quite a bit of work) invert this system, to obtain in $\mathcal{H}_{\hat{\alpha}}$ that for some sequence of local functions $(\phi_k^{i,+})_{k\in\mathbb{N}}$, in the limit $k\to\infty$ the identities

$$\limsup_{k\to\infty} \ll j_{0,e_i}^+ + d_s(\hat{\alpha})(\eta_{e_i}^+ - \eta_0^+) + D^+(\hat{\alpha})(\eta_{e_i} - \eta_0) + \mathscr{L}\phi_k^{i,+} \gg_{\hat{\alpha}} = 0,$$

The identification of the coefficients $d_s(\hat{\alpha})$ and $D^+(\hat{\alpha})$ as given in the introduction also follows from the explicit decomposition.

Remark 2 (Empty sites in the configuration) *As mentionned previously, because of the lack of mixing at high density, one needs to introduce in the gradients $\eta_{e_i}^+ - \eta_0^+$ indicator functions that vanish whenever there are not enough empty sites around $\{0, e_i\}$. This induces some significant technical diffucuity, but essentially does not change the scheme of proof above, since all the identities above hold as p is sent to ∞, where p is the size of the box in which empty sites need to be present. In order to give as clear a presentation as possible, we completely ignore this issue in these notes.*

5.3 Structure of the proof of the decomposition theorem for closed forms

We now sketch the proof of the decomposition

$$\mathfrak{C}_{\hat{\alpha}} = \overline{\mathfrak{E}_{\hat{\alpha}}} \oplus \mathfrak{J}, \tag{18}$$

which is at the center of the non-gradient method. The fact that the right-hand side is composed of germs of closed forms is straightforward, we focus on the reverse inclusion. Fix the germ of a closed form u, and also denote by u the corresponding (expanded) closed form.

We start by localising the problem, by considering on Σ_n the differential form

$$u_{x,x+z}^n = \mathbb{E}_{\nu_{\hat{\alpha}}}(u_{x,x+z} \mid \sigma(\hat{\eta}_x,\ x \in B_n)).$$

This is a closed form on B_n, and in particular according to Lemma 7 there exists a function φ_n such that
$$u^n_{x,x+z} = \nabla_{x,x+z}\varphi_n.$$
Considering the germ of an exact form $\frac{1}{(2n)^2}\nabla\Sigma_{\varphi_n}$, we can then write
$$\frac{1}{(2n)^2}\nabla\Sigma_{\varphi_n} = \frac{1}{(2n)^2}\sum_{x,x+e_i\in B_n}\tau_{-x}\nabla_{x,x+e_i}\varphi_n + \frac{1}{(2n)^2}\sum_{x\ or\ x+e_i\in B_n}\tau_{-x}\nabla_{x,x+e_i}\varphi_n,$$
because for any edge $(x, x+z)$ not intersecting B_n, the gradient $\nabla_{x,x+z}\varphi_n$ vanishes because φ_n is B_n-measurable.

The strategy is then straightforward : it is fairly easy to show that the bulk term (i.e. the first sum in the right-hand side) converges as $n\to\infty$ in $L^2(\nu_{\hat\alpha})$ to u_i, whereas the boundary term is, in the limit„ in the span of the j's. It is actually the estimation of the boundary terms that requires a sharp estimate on the spectral gap of the generator. To estimate the boundary terms however, the φ_n's need to be smoothed out, so that we replace them with
$$\widetilde{\varphi}_n = \mathbb{E}_{\nu_{\hat\alpha}}(\varphi_{3n} \mid B_n).$$
This is due to the fact that the boundary terms involve particle creation and deletion, instead of pure particle displacement.

We will not detail this decomposition, however proving that that $\nabla_{x,x+e_i}\varphi_n$ converges to u_i for any x in the bulk is just a consequence of the martingale convergence theorem. The proof that the boundary terms converge to a linear combination of the currents, on the other hand, is fairly more involved. We just comment here on the fact that the spectral gap is used to get some L^2 estimate on φ_n, and therefore on the boundary terms themselves. More precisely, one first uses the spectral gap, which yields
$$\mathbb{E}(\varphi_n^2) \leq Cn^2\mathbb{E}_{\nu_{\hat\alpha}}(\varphi_n\mathscr{L}\varphi_n).$$
Then, rewrite $\mathbb{E}_{\nu_{\hat\alpha}}(\varphi_n\mathscr{L}\varphi_n)$ as a sum of squares of gradients, then use both the translation invariance of $\nu_{\hat\alpha}$, and the fact that $\nabla_{x,x+z}\varphi_n$'s L^2 norm is that of u_i, to show that
$$\mathbb{E}(\varphi_n^2) \leq Cn^4\mathbb{E}_{\nu_{\hat\alpha}}(u_1^2+u_2^2).$$
This estimate is crucial to prove that the boundary terms are compact.

Remark 3 (Spectral gap and restriction to a subset of functions) *Here, we skip over another technical detail of the proof. The multicolor exclusion process does not have a priori a general spectral gap of the right order, which is crucial to Varadhan's non-gradient proof. However, Quastel [4] proves that once restricted to a class of functions depending linearly on the particle's types, the spectral gap becomes of the right order. Since both gradients and currents belong to this class of function, and since it is stable by the generator, this is actually enough to prove the decomposition of closed forms.*

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