Hydrodynamic limit for an active lattice gas

Swarming and phase transitions

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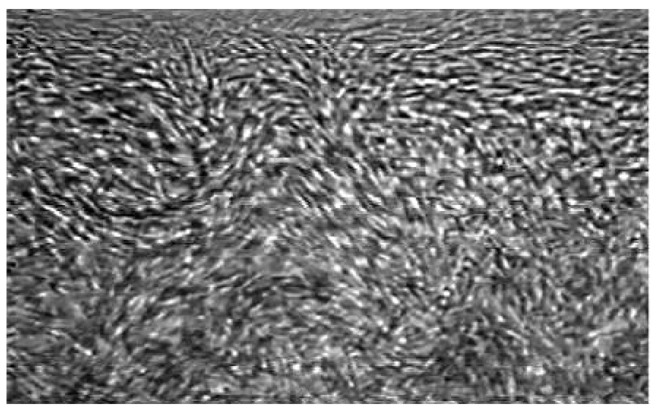

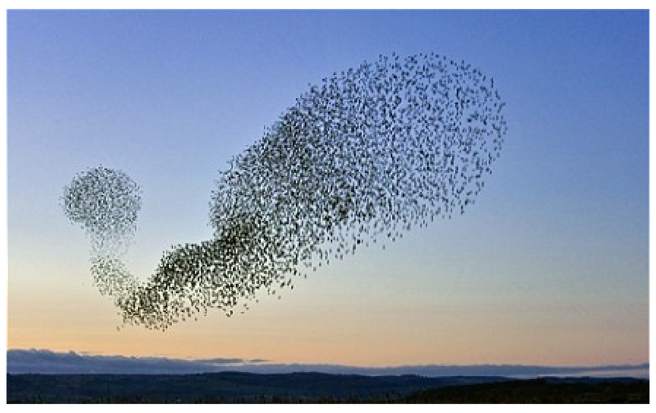


Abstract

Interest for the underlying mechanisms of collective animal motion has given rise in the last decades to an active and multidisciplinary field of study. Inspired by the pioneering work by Vicsek et al. [4], we consider a swarming model set on a lattice, in which each particle is characterized by an angular speed which orients the weakly asymmetric random walk it performs. Each particle also tend to align its angular speed with the average amongst its nearest neighbors. We obtain the scaling limit of this system as the span of the lattice goes to 0 under diffusive scaling, which requires the tools developed by Quastel and Varadhan for non-gradient dynamics, and poses several further technical and theoretical difficulties.

Introduction

Numerous examples of swarming and coherent motion can be observed in nature. A classical model for self-organized behavior is the Vicsek Model [4], which triggered a lot of interest for the modeling of animal collective behavior. The central question in the study of animal coherent motion is that of the phase transition between erratic individual motion, and global order, which arises as the strength of the alignment between individuals reaches a critical value (cf. Figure 1). Ample numerical evidence of such phase transitions have been obtained for *active matter models* (i.e. driven out-of-equilibrium by an energy influx at an individual level) in the last decades, and they are now fairly well understood from a physics standpoint. Mathematical proof of dynamical phase transitions, however, remain a challenge to this day.

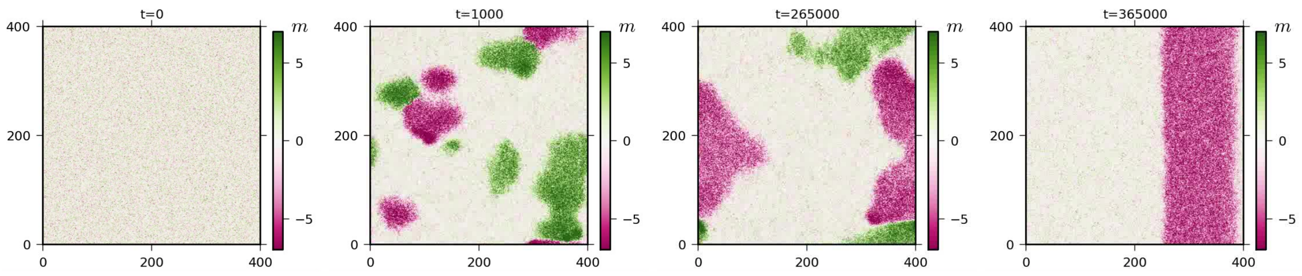

Figure 1: Emergence of global order for an alignment dynamics. From Solon & Tailleur, [3]

Apart from alignment-induced phase transitions, another phenomenon that can be observed with active matter models is the motility-induced phase separation (MIPS, cf. Figure 2), which occurs for models where the speed of each particle depends on the local density. If particles go slower in crowded areas, they tend to accumulate there, thus creating important density fluctuations.

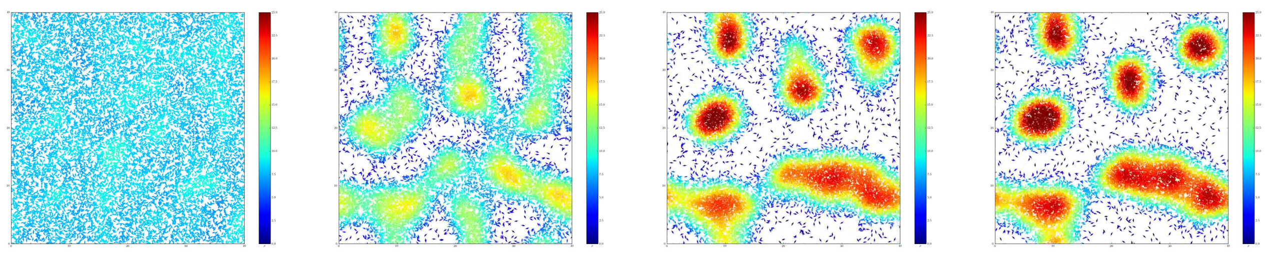

Figure 2: Congestion phenomenon for density-dependent particle speeds (MIPS). From Cates & Tailleur, [1]

We study a stochastic lattice gas with both a nearest-neighbor alignment dynamics and a congestion mechanism to observe a MIPS, and obtain its hydrodynamic limit.

Description of the model

The particle system we study evolves on the two-dimensional periodic domain $\mathbb{T}_N^2 = [\![0, N]\!]^2$. Each particle is characterized by an angle θ, and evolves according to weakly asymmetric random walk for which the asymmetry rate λ_i in the direction e_i is a function of θ. Only one particle can be simultaneously present at each site of the domain. Furthermore, the angle of each particle is updates at random times to align with the average angle among its neighbors.

Configurations of the model

- Each site of $\mathbb{T}_N^2$ is either *occupied by a particle with angle* θ ($\eta_x = 1$, and $\theta_x = \theta$), or *empty* ($\eta_x = 0$, and θ_x assumes the default value 0). The angle θ_x represent the favored motion direction for the particle at site x.
- We let $\widehat{\eta} = ((\eta_x, \theta_x))_{x\in\mathbb{T}_N^2}$ represent the complete *configuration*.

Infinitesimal Markov generator

The Markov generator of the process is given by $L_N = N^2\mathcal{L}^D + \mathcal{L}^G$, where $\mathcal{L}^D$ is the *displacement generator*, and $\mathcal{L}^G$ is the *alignment generator*, which actions are represented in Figure 3. For any local function f,

$$\mathcal{L}^D f(\widehat{\eta}) = \sum_{x\in\mathbb{T}_N} \sum_{\substack{\delta=\pm1\\ i=1,2}} \left(1 + \frac{\delta\lambda_i(\theta_x)}{N}\right) \eta_x \left(1 - \eta_{x+\delta e_i}\right) \left(f(\widehat{\eta}^{x,x+\delta e_i}) - f(\widehat{\eta})\right),$$

$$\mathcal{L}^G f(\widehat{\eta}) = \sum_{x\in\mathbb{T}_N} \int_0^{2\pi} c_{x,\beta}(\theta, \widehat{\eta})\eta_x \left(f(\widehat{\eta}^{x,\theta}) - f(\widehat{\eta})\right) d\theta,$$

- $\widehat{\eta}^{x,x+z}$ is the configuration where the particle in x has been moved to $x + z$
- $\widehat{\eta}^{x,\theta}$ is the configuration where θ_x has been set to θ
- $\lambda_i(\theta)$ represent the strength of the drift of a particle with angle θ in the direction e_i
- $c_{x,\beta}(\theta, \widehat{\eta})$ tunes the alignment dynamic.

The initial configuration is chosen at local equilibrium, and close to a smooth profile. We then consider a Markov process $(\widehat{\eta}(t))_{t\in[0,T]}$ driven by the generator L_N and starting from this initial configuration, which will be referred to as Active Exclusion Process (AEP).

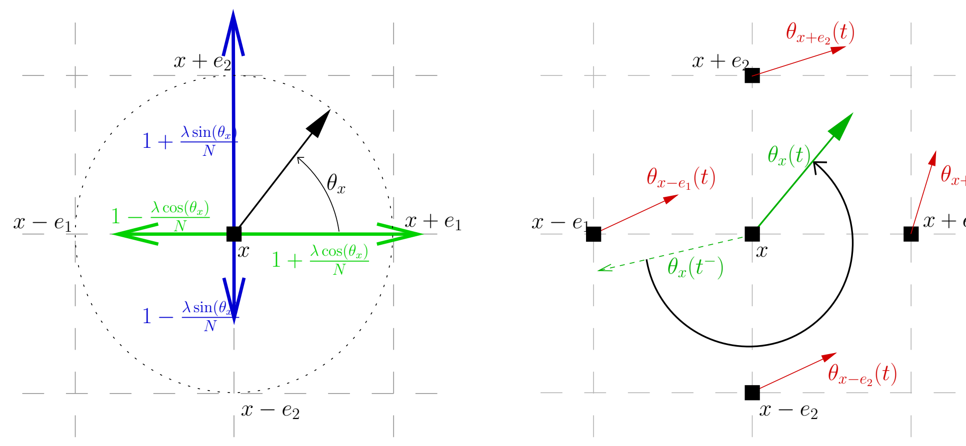

Figure 3: Description of the dynamics : displacement (left) and alignment (right).

Main result & elements of proof

We denote by $\widehat{\rho}_t(u, \theta)$ the macroscopic density of particles, which can be loosely characterized for any smooth function $H_t(u, \theta)$, by

$$\int_0^T \int_{\mathbb{T}} \int_0^{2\pi} \widehat{\rho}_t(u, \theta)H_t(u, \theta)d\theta du dt = \lim_{N\to\infty} \int_0^T \frac{1}{N^2} \sum_{x\in\mathbb{T}_N^2} \eta_x(t)H_t(x/N, \theta_x(t))dt.$$

Theorem 1 (Hydrodynamic limit for the AEP)**.** *Let $\rho_t(u) = \int_0^{2\pi} \widehat{\rho}_t(u, \theta)d\theta$ denote the local particle density, $\widehat{\rho}_t(u, \theta)$ is solution (in a weak sense) of the differential equation*

$$\partial_t\widehat{\rho}_t = \nabla.\left[\mathfrak{d}(\widehat{\rho}_t, \rho_t)\nabla\rho_t + d_s(\rho_t)\nabla\widehat{\rho}_t\right] + 2\nabla.\left[\mathfrak{s}(\widehat{\rho}_t, \rho_t)\lambda\overrightarrow{\Omega}_t + d_s(\rho_t)\widehat{\rho}_t \begin{pmatrix}\lambda_1(\theta)\\ \lambda_2(\theta)\end{pmatrix}\right] + \Gamma_t.$$

- *The quantity $\mathfrak{d}(\widehat{\rho}, \rho)$ is a diffusion coefficient for which we have an explicit expression*
- *$d_s(\rho)$ is the diffusion coefficient of a tagged particle for the SSEP at equilibrium with density ρ*
- *The quantity $\mathfrak{s}(\widehat{\rho}, \rho)$ is a conductivity coefficient linked to $\mathfrak{d}(\widehat{\rho}, \rho)$ by the Stokes-Einstein relation*
- *The local asymmetry $\overrightarrow{\Omega}_t$ is defined by*

$$\overrightarrow{\Omega}_t(u) = \int_{[0,2\pi[} \widehat{\rho}_t(u, \theta) \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} d\theta,$$

- *Γ_t is the local creation rate of particles with angle θ.*

Difficulties of the proof

Irreducibility : due to the exclusion rule, and to the multiple types of particles, The Active Exclusion Process loses its mixing properties as the density goes to 1 . This creates many technical difficulties throughout the proof. In particular, it becomes necessary to bound the particle density away from 1 at any time $t > 0$.

Non-gradient dynamics : the exclusion between particles with different angles prevents from expressing the instantaneous symmetric current as a discrete gradient. This is a major difficulty of the proof, and requires the tools developed by Varadhan and Quastel [2].

Out-of-equilibrium dynamics : The AEP is an *active matter model*, in which particles are affected by a weak drift. This constantly drives the system out of equilibrium, therefore some control is necessary to prove local equilibrium and to compare our process to its equilibrium counterpart for which explicit estimates can be obtained.

Conclusion & Research perspectives

The scaling limit of this active exclusion process could lead to proving a large deviations principle, and thus be the first step to unveil dynamical phase transitions for particle systems combining alignment and displacement.

References

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