DS1 - 2h - English version

October 13 2022

All random variables are defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Exercise 1 : Poisson Process

Let $\{N_t, \mathscr{F}_t; t \geq 0\}$ be a Poisson process of intensity $\lambda > 0$.
1) Show that $\{X_t := N_t - \lambda t, \mathscr{F}_t; t \geq 0\}$ is a martingale ;
2) Show that $\{X_t^2 - \lambda t, \mathscr{F}_t; t \geq 0\}$ is a martingale.

Exercise 2 : Submartingales

Let $\{X_t, \mathscr{F}_t; t \geq 0\}$ be a right-continuous submartingale and T be a stopping time. Show that $\{X_{T\wedge t}, \mathscr{F}_t; t \geq 0\}$ is a submartingale.
Hint. Admit the following fact : $X_{T\wedge t}$ is $\mathscr{F}_{T\wedge t}$-measurable, $\mathbb{E}[|X_{T\wedge t}|] < \infty$.

Exercise 3 : Bounded variation martingales

Let $\{M_t, \mathscr{F}_t; t \in [0, T]\}$ be a continuous, square integrable martingale. Suppose that $M_0 = 0$, $\mathbb{P}$-a.s.
1) If $\langle M\rangle_T = 0$, show that $M_t = 0$, $\forall\, t \in [0, T]$, $\mathbb{P}$-a.s.
2) A function $f : [0, T] \to \mathbb{R}$ is said to be of bounded variation if

$$V(f) := \sup_{\Pi:\,\text{partition of}\,[0,T]} \left\{ \sum_{k=1}^{m} \left| f(t_k) - f(t_{k-1}) \right| \right\} < \infty,$$

where $\Pi = \{0 = t_0 \leq \ldots \leq t_m = T\}$. If M is of bounded variation, $\mathbb{P}$-a.s., show that $M_t = 0$, $\forall\, t \in [0, T]$, $\mathbb{P}$-a.s.
3) Is the claim in Exercise 3.2 still true when M is not continuous ?

Exercise 4 : Brownian bridge

Let $\{W_t, \mathscr{F}_t^W; t \geq 0\}$ be a Wiener process (standard, 1-d Brownian motion, $W_0 = 0$). Fix $T > 0$ and define $B_t := E[W_t | W_T = 0]$, $\forall\, t \in [0, T]$.
1) What is the distribution of B_t for each $t \in [0, T]$?
2) Calculate $\operatorname{cov}(s, t) := \mathbb{E}[B_s B_t]$ for s, $t \in [0, T]$.
3) Show that $\{B_t; t \in [0, T]\}$ has a ($\mathbb{P}$-a.s.) continuous modification.

Solution of Exercise 1

For $0 \le s \le t$,

$$E[X_t \,|\, \mathscr{F}_s] = N_s + E[N_t - N_s \,|\, \mathscr{F}_s] - \lambda t$$

$$= N_s + \sum_{k=0}^{\infty} k \cdot \frac{[\lambda(t-s)]^k}{e^{\lambda(t-s)} k!} - \lambda t = N_s - \lambda s = X_s,$$

and

$$E[X_t^2 - \lambda t \,|\, \mathscr{F}_s] = E[N_t^2 \,|\, \mathscr{F}_s] - 2\lambda t E[N_t \,|\, \mathscr{F}_s] + \lambda^2 t^2 - \lambda t$$

$$\begin{aligned} &= N_s^2 + E[(N_t - N_s)^2 \,|\, \mathscr{F}_s] - 2\lambda s N_s + \lambda t(2\lambda s - \lambda t - 1) \\ &= N_s^2 + \sum_{k=0}^{\infty} k^2 \cdot \frac{[\lambda(t-s)]^k}{e^{\lambda(t-s)} k!} - 2\lambda s N_s + \lambda t(2\lambda s - \lambda t - 1) \\ &= N_s^2 - 2\lambda s N_s + \lambda^2 s^2 - \lambda s = X_s^2 - \lambda s. \end{aligned}$$

Solution of Exercise 2

Observe that $X_{T\wedge t}$ is $\mathscr{F}_{T\wedge t}$-measurable and $T \wedge t \le t$, it is also $\mathscr{F}_t$-measurable. Given $s \le t$, since $T \wedge t$, $T \wedge s$ are bounded stopping times, the optional sampling theorem yields that $E[|X_{T\wedge t}|] < \infty$ and

$$E[X_{T\wedge t} \,|\, \mathscr{F}_{T\wedge s}] \ge X_{T\wedge s}.$$

For any $A \in \mathscr{F}_s$,

$$A \cap \{T > s\} \cap \{T \wedge s \le t'\} = \begin{cases} \emptyset \in \mathscr{F}_{t'}, & t' < s, \\ A \cup \{T > s\} \in \mathscr{F}_{t'}, & t' \ge s, \end{cases}$$

so $A \cap \{T \ge s\} \in \mathscr{F}_{T\wedge s}$. Therefore,

$$\begin{aligned} E[X_{T\wedge t}\mathbf{1}_A] &= E[X_{T\wedge t}\mathbf{1}_{A\cap\{T\le s\}}] + E[X_{T\wedge t}\mathbf{1}_{A\cap\{T>s\}}] \\ &= E[X_{T\wedge s}\mathbf{1}_{A\cap\{T\le s\}}] + E[X_{T\wedge t}\mathbf{1}_{A\cap\{T>s\}}] \\ &\ge E[X_{T\wedge s}\mathbf{1}_{A\cap\{T\le s\}}] + E[X_{T\wedge s}\mathbf{1}_{A\cap\{T>s\}}] \\ &= E[X_{T\wedge s}\mathbf{1}_A]. \end{aligned}$$

In other words, $E[X_{T\wedge t} \,|\, \mathscr{F}_s] \ge X_{T\wedge s}$ for all $0 \le s \le t$.

Solution of Exercise 3

Observe that for $t \in [0, T]$, $0 \le \langle M \rangle_t \le \langle M \rangle_T = 0$, so $\langle M \rangle_t \equiv 0$. Recall that $M_t^2 - \langle M \rangle_t$ is a martingale, we obtain

$$\begin{aligned} 0 &= E\left[M_t^2 - M_0^2 \,|\, \mathscr{F}_0\right] = E\left[M_t^2 - (E[M_t|\mathscr{F}_0])^2\right] \\ &= E\left[(M_t - E[M_t|\mathscr{F}_0])^2 \,|\, \mathscr{F}_0\right]. \end{aligned}$$

Therefore, $M_t = E[M_t|\mathscr{F}_0] = M_0$, P-a.s.

Recall the definition of the quadratic variation

$$\langle M\rangle_T = P - \lim_{\|\Pi\|\to 0}\sum_{k=1}^m |M_{t_k} - M_{t_{k-1}}|^2.$$

Suppose that M is of bounded variation, then

$$\begin{aligned}\sum_{k=1}^m |M_{t_k} - M_{t_{k-1}}|^2 &\le \sup_{1\le k\le m}|M_{t_k} - M_{t_{k-1}}|\sum_{k=1}^m |M_{t_k} - M_{t_{k-1}}| \\ &\le V(M_\cdot)\sup_{1\le k\le m}|M_{t_k} - M_{t_{k-1}}|.\end{aligned}$$

From the continuity of M, the right-hand side above vanishes almost surely as $\|\Pi\| \to 0$, so $\langle M\rangle_T = 0$.

The conclusion fails when M is not continuous, for instance $M_t := N_t - \lambda t$ where N_t is a Poisson process.

Solution of Exercise 4

For $t \in [0, T]$, the joint density of (W_t, W_T) is

$$f_{t,T}(s, y) = p(t; 0, x)p(T - t; x, y), \quad p(t; x, y) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}.$$

Therefore, the density of $B_t := E[W_t|W_T = 0]$ is given by

$$\frac{p(t; 0, x)p(T - t; x, 0)}{\int_{\mathbb{R}} p(t; 0, x)p(T - t; x, 0)dx} = \frac{\sqrt{T}}{2\pi\sqrt{t(T - t)}}\exp\left\{-\frac{Tx^2}{2t(T - t)}\right\}.$$

So B_t is a Gaussian variable with mean 0 and variance $\frac{t(T-t)}{T}$.

Similarly as above, for $s < t \le T$, the joint density of (B_s, B_t) is

$$\frac{p(s; 0, x)p(t - s; x, y)p(T - t; y, 0)}{\iint_{\mathbb{R}^2} p(s; 0, x)p(t - s; x, y)p(T - t; y, 0)dxdy}.$$

Notice that

$$\begin{aligned}p(s; 0, x)p(t - s; x, y)p(T - t; y, 0) &= \frac{1}{(2\pi)^{3/2}\sqrt{s(t - s)(T - t)}} \\ &\exp\left\{-\frac{\frac{T-t}{t}(tx - sy)^2 + \frac{s(t-s)T}{t}y^2}{2s(t - s)(T - t)}\right\}.\end{aligned}$$

So that $tB_s - sB_t$ and B_t are Gaussian and they are independent.

$$E[B_sB_t] = E\left[B_s - \frac{s}{t}B_t\right]E[B_t] + \frac{s}{t}E[B_t^2] = \frac{s}{t}E[B_t^2] = s - \frac{st}{T}.$$

Considering also the case $s \ge t$, $\mathrm{cov}(s, t) = s \wedge t - \frac{st}{T}$.

For the continuous modification, just note that for $s < t \leq T$,

$$E[(B_t - B_s)^2] = \frac{t(T-t)}{T} + \frac{s(T-s)}{T} - 2s + \frac{2st}{T} = t - s - \frac{(t-s)^2}{T}.$$

As $B_t - B_s$ ia Gaussian,

$$E[(B_t - B_s)^4] = C\left[t - s - \frac{(t-s)^2}{T}\right]^2 \leq C(t-s)^2.$$

One can apply Kolmogorov continuity theorem.