Advanced Probability
Master 2 Sciences, Mention Math.
Parcours recherche

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DS2 - 2h - English version

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- *All random variables are defined on some probability space* $(\Omega, \mathscr{F}, \mathbb{P})$.
- *The exercises are independent, take a look at every exercise before choosing which ones to tackle first.*
- *The points for each exercise are given as an indication and may be subject to changes. It is not necessary to do everything to obtain the maximal grade.*

Exercise 1 : Random walks on $\mathbb{N}$, ~ 10pts

We consider a continuous time random walk X_t on $\mathbb{N}$, jumping from x to $x+1$ at rate $p \in]0,1[$ for $x \in \mathbb{N}$, and jumps at rate $q := 1-p$ from x to $x-1$ for $x \in \mathbb{N} \setminus \{0\}$.

1) Write down the intensity matrix for this Markov process, and give its graphic representation. What are the communicating classes for this process ?

2) We are interested in the stationary states for the Markov process. Assume that X_t admits a stationary probability measure, denoted μ.

(i) What is the support of μ ? Justify that $p\mu(0) = q\mu(1)$ and for any $x = \{1, 2, \ldots\}$

$$p(\mu(x-1) - \mu(x)) = q(\mu(x) - \mu(x+1)).$$

3) In the three cases below, determine whether there exists a reversible state, a stationary state, and if so, define it. Interpret the result in terms of the long-time behavior of $(X_t)_{t\geq 0}$.

(i) $p = 1/2$ (ii) $p > 1/2$ (iii) $p < 1/2$.

4) For $k \in \mathbb{N}$, we denote by $\mathbb{P}_k$ the distribution of $(X_t)_{t\geq 0}$ started from $X_0 = k$, and $\mathbb{E}_k$ the corresponding expectation. We define T_k the first time X_t hits k. Fix $K \in \mathbb{N}$, and $n \in \mathbb{N}$, compute

$$g_n = \mathbb{P}_n(T_K \leq T_0).$$

Hint : find an equation satisfied by g_n.

5) We now assume that $p = 1$. What are the communicating classes ? For $X_0 = 0$, what is the distribution of the process $(X_t)_{t\geq 0}$?

<u>SOLUTION</u> :

1)

$$L = \begin{pmatrix} -p & p & & & \\ q & -p-q & p & & (0) \\ & q & -p-q & \ddots & \\ & (0) & \ddots & \ddots & \\ & & & & \end{pmatrix}$$

Since p is different from 0 and 1, the only communicating class is $\mathbb{N}$.

2) The support of μ is $\mathbb{N}$. For any function f, we must have in matrix form $\mathbb{E}_\mu(Lf) = \mu Lf = 0$ (μ being a row vector) by definition, so that by choosing $f(y) = \mathbf{1}_{y=0}$ we obtain the result. The second identity is obtained by choosing $f(y) = \mathbf{1}_{y=x}$.

3) (i) If $p = 1/2$, the random walk is symmetric, and therefore far from the origin, it should have the same probability to be everywhere, so that the latter must be 0. There is no reversible probability measure, and the only stationary measure is 0. Indeed, the previous identity yields that the discrete laplacian of μ is 0, so that μ must be linear. The only linear function with bounded integral over $\mathbb{N}$ is 0.

(ii) If p is larger than $1/2$, there is once again no reversible probability measure, since the only reversible measure is identically 0. Since the random walk drifts right, the random walker escapes to infinity, and the unique stationary measure is also identically 0. To prove the latter, write $g_x = \mu(x+1) - \mu(x)$ the discrete gradient of μ, the last identity yields for $x \geq 0$

$$pg_{x-1} = qg_x \qquad \Rightarrow \qquad g_x = \left(\frac{p}{q}\right)^x g_0.$$

In particular,

$$\mu_x = \mu_0 + \sum_{y=0}^{x-1} g_y = \mu_0 + g_0 \frac{1-(p/q)^x}{1-p/q}$$

μ_x diverges except if $g_0 = 0$, but then μ is constant and must be 0.

(iii) Finally, if $p < q$, there is a reversible measure, which is then also stationary, and is given by $p\mu_x = q\mu_{x+1}$, so that

$$\mu_x = \mu_0 (p/q)^x.$$

Since μ must be a probability measure, we find $\mu_0 = 1 - p/q$.

4) Clearly $g_K = 1$, and $g_0 = 0$. By Markov property,

$$g_k = qg_{k+1} + pg_{k-1},$$

so that

$$\frac{p}{q}(g_k - g_{k-1}) = g_{k+1} - g_k.$$

Let $h_k = g_{k+1} - g_k$, we obtain $h_k = h_0 (p/q)^k$, therefore

$$g_k = g_0 + \sum_{n=0}^{k-1} h_n = h_0 \frac{1-(p/q)^k}{1-p/q}.$$

Since $g_K = 1$, we obtain

$$g_k = \frac{1-(p/q)^k}{1-(p/q)^K}.$$

5) If $p = 1$, no state is accessible from a larger state, so that the communicating classes are the $\{x\}$, for every $x \in \mathbb{N}$. X_t increases by one at rate one, so that $(X_t)_{t\geq 0}$ is a rate 1 Poisson process. □

Exercise 2 : LDP for exponential and Poisson variables, ~ 10pts

1) We consider first an i.i.d. sequence of Poisson variables $P_k \sim Poi(\lambda)$.

(i) Compute the log-MGF $\Lambda_P(t)$ of P_1.

(ii) After justifying its existence, compute depending on the value of $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{k=1}^{n} P_k \geq nx\right).$$

2) We now consider an i.i.d. sequence of exponential variables $E_k \sim Exp(\lambda)$.

(i) Compute the log-MGF $\Lambda_E(t)$ of E_1.

(ii) After justifying its existence, compute for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{k=1}^{n} E_k \geq nx\right).$$

3) (i) Considering a rate λ Poisson process, show that for any $n \in \mathbb{N}$, $t > 0$,

$$\mathbb{P}\left(Poi(\lambda t) \geq n\right) = \mathbb{P}\left(\sum_{k=1}^{n} E_k \leq t\right).$$

(ii) For $x > 0$, define $P^x \sim Poi(\lambda x)$, justify that for any $x > 0$,

$$\Lambda^\star_{P^x}(1) = x\Lambda^\star_P(1/x).$$

Deduce from the previous questions an identity between the Legendre transforms $\Lambda^\star_P$ and $\Lambda^\star_E$ of Poisson and exponential variables, and verify it on the answers to questions 1) and 2).

<u>Solution</u> :

1) (i) We compute for $t \in \mathbb{R}$

$$\log \mathbb{E}(e^{tP_1}) = \log\left[\sum_{k\geq 0} \frac{(e^t\lambda)^k}{k!} e^{-\lambda}\right] = \lambda(e^t - 1).$$

(ii) The log-MGF is finite everywhere, we can apply Cramér's theorem once we have computed its Legendre transform.

$$\Lambda^\star_P(x) = \sup_t\{xt - \lambda(e^t - 1)\}.$$

To compute the latter, derive in t, to obtain for $x > 0$

$$x_t = \frac{d}{dt}[\lambda(e^t - 1)] = \lambda e^t \qquad \Rightarrow \qquad t_x = \log(x/\lambda),$$

so that, i.e.

$$\Lambda_P^\star(x) = x\log(x/\lambda) + \lambda - x.$$

for $x \geq 0$, and $+\infty$ otherwise, with the convention $0\log 0 = 0$. By Cramér's theorem, we therefore have for any $x > \lambda$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{k=1}^{n} P_k \geq nx\right) = \begin{cases} -\Lambda_P^\star(x) & \text{for } x > \lambda = \mathbb{E}(P_1) \\ 1/2 & \text{by CLT for } x = \lambda \\ 0 & \text{for } x < \lambda \end{cases}$$

2) Similarly,

$$\Lambda_E^\star(t) := \log \mathbb{E}(e^{tE_1}) = \log \int_{x\geq 0} \lambda e^{-\lambda x + tx} dx = \begin{cases} +\infty & \text{for } t \geq \lambda \\ \log \frac{\lambda}{\lambda - t} & \text{for } t < \lambda \end{cases}.$$

(i) Once again, for positive λ, the $\log$-MGF is finite around 0, so that we can apply Cramér's theorem once we compute the Legendre transform. we find

$$x_t = \frac{d}{dt}\Lambda_E^\star(t) = \frac{1}{\lambda - t} \qquad \Rightarrow \qquad t_x = \lambda - \frac{1}{x},$$

so that

$$\Lambda_E^\star(x) = \lambda x - 1 - \log(\lambda x) \qquad \text{for } x \geq 0,$$

and $\Lambda_E^\star(x) = +\infty$ otherwise. Once again, By Cramér's theorem, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\sum_{k=1}^{n} E_k \geq nx\right) = \begin{cases} -\Lambda_E^\star(x) & \text{for } x > 1/\lambda = \mathbb{E}(E_1) \\ 1/2 & \text{by CLT for } x = 1/\lambda \\ 0 & \text{for } x < 1/\lambda \end{cases}$$

3) (i) A Poisson process increases by 1 at rate Λ, so that letting $\tau_k \sim Exp(\lambda)$ be its holding times, we can rewrite

$$\mathbb{P}\left(Poi(\lambda t) \geq n\right) = \mathbb{P}\left(\sum_{k=0}^{n-1} \tau_k \leq t\right).$$

Denoting $E_k = \tau_{k-1}$, which are i.i.d. $Exp(\lambda)$, proves the identity.

(ii) We define $t = xn$, and note that in distribution, $Poi(\lambda xn) = \sum_{k=1}^{n} P_k'$ where $P_k^x \sim Poi(\lambda x)$, therefore

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} P_k^x \geq 1\right) = \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} E_k \leq x\right)$$

therefore

$$\frac{1}{n}\log \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} P_k^x \geq 1\right) = \frac{1}{n}\log \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} E_k \leq x\right),$$

so that for $x > \lambda$, $1/x < 1/\lambda$, by Cramér's Theorem,

$$\Lambda_{P^x}^\star(1) = x\Lambda_P^\star(1/x) = \Lambda_E^\star(x).$$

The same is true for $x < \lambda$. □

Exercise 3 : Continuity and σ-algebras, $\sim$ 6pts

Let $X = \{X_t; t \geq 0\}$ be a real-valued stochastic process defined on some probability space $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_t^X; t \geq 0\}$ be the natural filtration of X. Show the followings.

1. If X is left-continuous, then $\{\mathscr{F}_t^X\}$ is left-continuous.
2. If X is right-continuous and $\tau := \inf\{t \geq 0; X_t > 1\}$, then

$$\{\tau < t\} \in \mathscr{F}_t^X, \quad \forall\, t \geq 0.$$

Hint. Recall that the natural filtration is defined as

$$\mathscr{F}_t^X := \sigma(X_s; s \in [0,t]), \quad \forall\, t \geq 0.$$

<u>*Solution*</u> :

1) Fix some arbitrary $t \geq 0$. It suffices to show that

$$\mathscr{F}_{t-}^X := \sigma\Big(\bigcup_{0 \leq s < t} \mathscr{F}_s^X\Big) = \mathscr{F}_t^X.$$

Observe that $\mathscr{F}_{t-}^X \subseteq \mathscr{F}_t^X$ is straightforward. On the other hand,

$$\Big\{X_{t-\frac{1}{n}} \leq x\Big\} \in \mathscr{F}_{t-\frac{1}{n}}^X \subseteq \mathscr{F}_{t-}^X, \quad \forall\, x \in \mathbb{R}, n \in \mathbb{N}_+,$$

so $X_{t-\frac{1}{n}}$ is $\mathscr{F}_{t-}^X$-measurable. Since $t \mapsto X_t$ is left-continuous,

$$X_t = \lim_{n\to\infty} X_{t-\frac{1}{n}} \text{ is } \mathscr{F}_{t-}^X\text{-measurable.}$$

Therefore, $\mathscr{F}_t^X \subseteq \mathscr{F}_{t-}^X$ and the equality follows.

2) By the definition of τ,

$$\{\tau < t\} = \bigcup_{s \in [0,t)} \{X_s > 1\}.$$

Denote by $\mathbb{Q}$ the set of rational numbers. We show that

$$\bigcup_{s \in [0,t)} \{X_s > 1\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s > 1\}.$$

Indeed, suppose that $X_s > 1$ for some $s \in [0,t)$, then $\exists\, s_n \in \mathbb{Q}$ such that $s < s_n < t$ and $\lim_{n\to\infty} s_n = s$. Since $s \mapsto X_s$ is right-continuous,

$$\lim_{n\to\infty} X_{s_n} = X_s > 1.$$

Then, $\exists$ sufficiently large n such that $X_{s_n} > 1$. Therefore,

$$\bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s > 1\} \subseteq \bigcup_{s \in [0,t)} \{X_s > 1\} \subseteq \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s > 1\}.$$

The equality then follows. As each $\{X_s > 1\} \in \mathscr{F}_s^X \subseteq \mathscr{F}_t^X$,

$$\{\tau < t\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s > 1\} \in \mathscr{F}_t^X.$$

□

Exercise 4 Reflected Brownian motion, ~ 6pts

Let $\{B_t; t \geq 0\}$ be a standard, one-dimensional Brownian motion such that $B_0 = 0$.

1. For $n \in \mathbb{N}_+$ and $0 < t_1 < \ldots < t_n$, compute the joint probability density function of $(B_{t_1}, \ldots, B_{t_n})$.
2. Define $\{W_t; t \geq 0\}$ by reflecting B_t at -1 :

$$W_t(\omega) := \begin{cases} B_t(\omega), & \text{if } B_t(\omega) \geq -1, \\ -2 - B_t(\omega), & \text{if } B_t(\omega) < -1. \end{cases}$$

Compute the probability density function of W_t.

Solution :

1) For $(a_1, \ldots, a_n) \in \mathbb{R}^n$ denote

$$A = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_j < a_j, \forall\, j = 1, \ldots, n\}.$$

Fix $x_0 = 0$ and define the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x_1, \ldots, x_n) = (x_1 - x_0, \ldots, x_n - x_{n-1}),$$

The Jacobian determinant $|\det D| = 1$, where

$$D = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

Let $t_0 = 0$. By the definition of Brownian motion,

$$P((B_{t_1}, \ldots, B_{t_n}) \in A) = P(\Phi(B_{t_1}, \ldots, B_{t_n}) \in \Phi(A))$$
$$= \int_{\Phi(A)} \prod_{j=1}^{n} p(t_j - t_{j-1}; y_j) dy_1 \ldots dy_n, \quad p(t; y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}.$$

Applying the change of variables $(y_1, \ldots, y_n) = \Phi(x_1, \ldots, x_n)$,

$$P(B_{t_1} < a_1, \ldots, B_{t_n} < a_n)$$

$$= \int_A \prod_{j=1}^n p(t_j - t_{j-1}; x_j - x_{j-1}) |\det D| \, dx_1 \ldots dx_n,$$

Recalling that $|\det D| = 1$, the joint probability density function is

$$\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left\{-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right\}.$$

2) For any $[a, b] \subseteq [-1, \infty)$,

$$\begin{aligned} P(W_t \in [a, b]) &= P(B_t \in [a, b] \cup [-2 - b, -2 - a]) \\ &= \frac{1}{\sqrt{2\pi t}} \left(\int_a^b + \int_{-2-b}^{-2-a} \right) e^{-\frac{x^2}{2t}} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_a^b \left[e^{-\frac{x^2}{2t}} + e^{-\frac{(x+2)^2}{2t}} \right] dx. \end{aligned}$$

Hence the probability density function of W_t is

$$f(x) = \frac{\mathbf{1}_{\{x \geq -1\}}}{\sqrt{2\pi t}} \left(\exp\left\{-\frac{x^2}{2t}\right\} + \exp\left\{-\frac{(x+2)^2}{2t}\right\} \right).$$

□