Master 2 Mathématiques
Parcours recherche

TD1 – Markov jump processes*

Exercise 1 : Explosion of birth processes

Fix a sequence of non-negative rates $(\lambda_j)_{j\in\mathbb{N}}$. A birth process is a Markov jump process on $\mathbb{N}$ with rates

$$\ell_{j,j+1} = \lambda_j \quad \text{and} \quad \ell_{j,j} = -\lambda_j,$$

started from $X_0 = 1$.

1) For $k \geq 1$, we denote by S_k the time of the process' $k-1$-th jump, $S_k = \inf\{t, X_t = k\}$, with $S_1 := 0$, and we denote by $S_\infty = \sup_k S_k = \lim_{k\to\infty} S_k$ its explosion time. Compute $\mathbb{E}(S_\infty)$, and deduce a criterion for non-explosion in finite time for the markov chain.

2) We now want to prove that if $\sum \frac{1}{\lambda_j} = \infty$, the chain a.s. does not explode in finite time, $\mathbb{P}(S_\infty < \infty) = 0$.

(i) Fix a non-negative sequence $(\alpha_k)_{k\in\mathbb{N}}$, after proving that $\forall x > -1$,

$$\frac{x}{1+x} \leq \log(1+x) \leq x$$

show that

$$\prod_{k=1}^{\infty} \frac{1}{1+\alpha_k} = 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \alpha_k = \infty.$$

(ii) Compute for $k \geq 2$

$$\mathbb{E}\left(\exp(-(S_k - S_{k-1}))\right).$$

(iii) Deduce from it $\mathbb{E}(\exp(-S_\infty))$, and conclude.

Exercise 2 : A simple birth process

Fix $\lambda > 0$, we consider a birth process (see previous exercise) with rate $\lambda_j = \lambda j$, started from $X_0 = 1$. Define $p_j(t) = \mathbb{P}(X_t = j)$.

1) Write down Kolmogorov forward equations for $p_1(t)$ and $p_j(t)$, $j \geq 2$.

2) Show that X_t follows a geometric distribution with parameter $e^{-\lambda t}$. Does the process explode in finite time ?

*For any typo/question, please contact me at *clement.erignoux@inria.fr*. The exercise sheets will be put on the webpage, http://chercheurs.lille.inria.fr/cerignou/homepage.html in the "teaching" section.

3) Compute the expected population size at time t.

Exercise 3 : A simple Markov process

Fix $\alpha, \beta > 0$, and consider a Markov process on $E = \{1, 2\}$, with generator matrix

$$L = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

1) (i) Give a graphic representation of the Markov process.
(ii) Write down the Kolmogorov equation for $P_t[1, 1]$.
(iii) Prove the identity

$$P_t'[1, 1] + (\alpha + \beta)P_t[1, 1] = \beta.$$

(iv) Solve this equation to determine $P_t[1, 1]$.

2) (i) Compute L^2 as a function of L.
(ii) Deduce a simple formula for L^n.
(iii) Deduce from the previous questions that

$$P_t = I_2 + \frac{L}{\alpha + \beta}(1 - e^{-(\alpha+\beta)t}).$$

(iv) Check the result obtained at the previous question.

3) Does this chain have an invariant measure ?

Exercise 4 : A Markov triangle

We consider a Markov process X_t on a triangle, with vertices 1, 2, 3 going clockwise. In a small timestep dt, the process moves one step clockwise with probability $\alpha dt + O(dt^2)$, and counter clockwise with probability $\beta dt + O(dt^2)$, otherwise it stays put.

1) (i) Give the intensity matrix for this process, and give a graphic representation.
(ii) What is the probability starting from state 1, to hit state 2 before state 3 ?
(iii) We denote by T^3 the first time the process hits state 3. Compute $\mathbb{E}_1(T^3)$, which is the expectation starting from state 1 of the hitting time of state 3.

2) (i) What is the transition matrix of the skeleton (Y_k) of the process ?
(ii) What is the probability, for the skeleton, to hit state 2 before state 3, starting from state 1 ?
(iii) What is the expected number of steps in the skeleton to hit state

3 starting from state 1 ?

Exercise 5 : A queue process

At the post office, clients arrive at a single counter and exit as they are taken care of. We assume that at a rate λ (i.e. at the ringing times of a rate λ Poisson clock), a random number of clients (independant from the poisson clock and from the other arrivals) arrives in the queue, according to a distribution $\mathbf{p} := (p_k)_{k\in\mathbb{N}}$ on $\mathbb{N}$. We assume that $p_0 = 0$. All clients are in a single line, and the first in the line is taken care of at a rate $\gamma > 0$.

1) Write down the intensity matrix for this process, and justify that the total number of clients N_t that have entered the post office before time t can be written

$$N_t = \sum_{k=1}^{Q_t} \xi_k,$$

where Q_t follows a Poisson distribution with a parameter that will be indicated, and the (ξ_k)'s are i.i.d. variables with distribution $\mathbf{p}$.

2) (i) Justify that the number M_t of people in the post office is an irreducible Markov process, and its unique communicating class is $\mathbb{N}$.

(ii) Show that M_t does not explode in finite time.

3) Assume that μ is a stationary probability distribution for M. We define $g(s) = \sum_k p_k s^k$ (resp $\psi(s) = \sum_k \mu_k s^k$) the probabillity generating function of $\mathbf{p}$. (resp. μ).

(i) Write down the equations satisfied by μ, and show that

$$\psi(s) = \frac{\gamma\mu_0(1-s)}{s\lambda(g(s)-1)+\gamma(1-s)}.$$

(ii) Let $m = \sum kp_k$ be the average of $\mathbf{p}$. Show that if there exists an invariant probability measure, we must have $m < \infty$ and $m\lambda/\gamma < 1$

(iii) Show that if $m\lambda/\gamma < 1$ there exits a unique invariant probability measure μ.

(iv) Assume $m\lambda/\gamma < 1$, compute $\mathbb{E}_\mu(X_t)$.