Master 2 Mathématiques
Parcours recherche

TD1 – Markov jump processes*

Exercise 1 : Explosion of birth processes

Fix a sequence of non-negative rates $(\lambda_j)_{j\in\mathbb{N}}$. A birth process is a Markov jump process on $\mathbb{N}$ with rates

$$\ell_{j,j+1} = \lambda_j \quad \text{and} \quad \ell_{j,j} = -\lambda_j,$$

started from $X_0 = 1$.

1) For $k \geq 1$, we denote by S_k the time of the process' $k-1$-th jump, $S_k = \inf\{t, X_t = k\}$, with $S_1 := 0$, and we denote by $S_\infty = \sup_k S_k = \lim_{k\to\infty} S_k$ its explosion time. Compute $\mathbb{E}(S_\infty)$, and deduce a criterion for non-explosion in finite time for the markov chain.

2) We now want to prove that if $\sum \frac{1}{\lambda_j} = \infty$, the chain a.s. does not explode in finite time, $\mathbb{P}(S_\infty < \infty) = 0$.

(i) Fix a non-negative sequence $(\alpha_k)_{k\in\mathbb{N}}$, after proving that $\forall x > -1$,

$$\frac{x}{1+x} \leq \log(1+x) \leq x$$

show that

$$\prod_{k=1}^{\infty} \frac{1}{1+\alpha_k} = 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \alpha_k = \infty.$$

(ii) Compute for $k \geq 2$

$$\mathbb{E}\left(\exp(-(S_k - S_{k-1}))\right).$$

(iii) Deduce from it $\mathbb{E}(\exp(-S_\infty))$, and conclude.

Answer :

1) We wite $S_\infty = \sum_k S_{k+1} - S_k$, whose expectation is therefore $\sum_k \lambda_k^{-1}$. If the sum is finite, then $S_\infty < \infty$ with probability 1, and therefore the chain is explosive. To be non-explosive, we must have $\sum_k \lambda_k^{-1} = \infty$.

2) We now assume that the sum is finite, and prove that a.s. the birth process does not explode in finite time. For x positive,

$$\frac{1}{(1+x)^2} \leq \frac{1}{1+x} \leq 1$$

*For any typo/question, please contact me at *clement.erignoux@inria.fr*. The exercise sheets will be put on the webpage, http://chercheurs.lille.inria.fr/cerignou/homepage.html in the "teaching" section.

and for x negative the inequalities are in the other direction. It is then sufficient to integrate between 0 and x to obtain the wanted bound.

Now, write

$$\exp\left(-\sum_{k=1}^{\infty}\alpha_k\right) \le \prod_{k=1}^{\infty}\frac{1}{1+\alpha_k} = \exp\left(-\sum_{k=1}^{\infty}\log(1+\alpha_k)\right) \le \exp\left(-\sum_{k=1}^{\infty}\frac{\alpha_k}{1+\alpha_k}\right).$$

The left-hand side proves one implication. For the other, write

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{1+\alpha_k} = \sum_{k=1}^{\infty}\frac{\alpha_k}{1+\alpha_k}\mathbf{1}_{\{\alpha_k>1\}} + \sum_{k=1}^{\infty}\frac{\alpha_k}{1+\alpha_k}\mathbf{1}_{\{\alpha_k\le 1\}} \ge \frac{1}{2}\sum_{k=1}^{\infty}\mathbf{1}_{\{\alpha_k>1\}} + \frac{1}{2}\sum_{k=1}^{\infty}\alpha_k\mathbf{1}_{\{\alpha_k\le 1\}}.$$

If $\sum_{k=1}^{\infty}\alpha_k = \infty$, one of those two sums is infinite, which proves the converse implication.

(i) The expectation is equal to

$$\lambda_k\int_0^{+\infty} e^{-s}e^{-\lambda_k s}ds = \frac{\lambda_k}{1+\lambda_k} = \frac{1}{1+\alpha_k},$$

with $\alpha_k = 1/\lambda_k$.

(ii) Since the successive holding times are independant, we have

$$\mathbb{E}(\exp(-S_\infty)) = \prod_k \mathbb{E}\left(\exp(-(S_k - S_{k-1}))\right) = \prod_k \lambda_k\int_0^{+\infty} e^{-s}e^{-\lambda_k s}ds = \prod_k\frac{1}{1+\alpha_k}$$

In particular, according to the previous questions,

$$S_\infty = \infty \text{ a.s.} \quad\Leftrightarrow\quad \exp(-S_\infty) = 0 \text{ a.s.} \quad\Leftrightarrow\quad \mathbb{E}(\exp(-S_\infty)) = 0 \quad\Leftrightarrow\quad \sum_{k=1}^{\infty}\alpha_k = \infty$$

□

Exercise 2 : A simple birth process

Fix $\lambda > 0$, we consider a birth process (see previous exercise) with rate $\lambda_j = \lambda j$, started from $X_0 = 1$. Define $p_j(t) = \mathbb{P}(X_t = j)$.

1) Write down Kolmogorov forward equations for $p_1(t)$ and $p_j(t)$, $j \ge 2$.

2) Show that X_t follows a geometric distribution with parameter $e^{-\lambda t}$. Does the process explode in finite time ?

3) Compute the expected population size at time t.

Answer :

1) $p_1'(t) = -\lambda p_1(t)$, and

$$p_j'(t) = \lambda(j-1)p_{j-1}(t) - \lambda j p_j(t).$$

2) It is straightforward to show that $p_1(t) = e^{-\lambda t}$. Assume that for any $k \leq j-1$, and any t $p_k(t) = e^{-\lambda t}(1-e^{-\lambda t})^{k-1}$. By Markov property, for $j \neq 1$, since the evolution of a birth process started from two can be split into two independant birth processes,

$$
\begin{aligned}
p_j(t) &= \int_0^t \lambda e^{-\lambda s} \sum_{k=1}^{j-1} p_k(t-s)p_{j-k}(t-s)ds \\
&= (j-1)\lambda e^{-\lambda t} \int_0^t e^{-\lambda(t-s)}(1-e^{-\lambda(t-s)})^{j-2} ds \\
&= (j-1)e^{-\lambda t} \int_{e^{-\lambda t}}^1 (1-u)^{j-2} du \\
&= (j-1)e^{-\lambda t} \int_0^{1-e^{-\lambda t}} u^{j-2} du \\
&= e^{-\lambda t}(1-e^{-\lambda t})^{j-1}
\end{aligned}
$$

The probability that the process explodes in finite time is

$$\mathbb{P}(\exists t < \infty, X_t = \infty) \leq \sum_{k=0}^{+\infty} \mathbb{P}(X_k = \infty) = 0.$$

3) It is the expectation of a geometric variable with parameter $e^{-\lambda t}$, therefore $\mathbb{E}(X_t) = e^{\lambda t}$. □

Exercise 3 : A simple Markov process

Fix $\alpha, \beta > 0$, and consider a Markov process on $E = \{1, 2\}$, with generator matrix

$$L = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

1) (i) Give a graphic representation of the Markov process.
   (ii) Write down the Kolmogorov equation for $P_t[1,1]$.
   (iii) Prove the identity

$$P_t'[1,1] + (\alpha+\beta)P_t[1,1] = \beta.$$

   (iv) Solve this equation to determine $P_t[1,1]$.

2) (i) Compute L^2 as a function of L.
   (ii) Deduce a simple formula for L^n.
   (iii) Deduce from the previous questions that

$$P_t = I_2 + \frac{L}{\alpha+\beta}(1-e^{-(\alpha+\beta)t}).$$

   (iv) Check the result obtained at the previous question.

3) Does this chain have an invariant measure ?

ANSWER :

1) (i) Easy !

(ii) $P_t' = LP_t$, so that $P_t[1,1]' = -\alpha P_t[1,1] + \beta P_t[1,2]$

(iii) Write $P_t[1,1] = 1 - P_t[1,2]$ to prove the identity.

(iv)

$$P_t[1,1] = \frac{\beta}{\alpha+\beta} + Ae^{-(\alpha+\beta)t},$$

and the initial condition yields $A = \alpha/(\alpha+\beta)$.

2) (i)

$$L^2 = -(\alpha+\beta)L$$

(ii)

$$L^n = (-\alpha-\beta)^{n-1}L$$

(iii)

$$P_t = e^{tL} = I_2 + t\sum_{n=0}^{+\infty} \frac{(-t)^{n-1}(\alpha+\beta)^{n-1}}{n!}L = I_2 + \frac{1}{\alpha+\beta}\left(1 - e^{-t(\alpha+\beta)}\right)L,$$

on retrouve bien

$$P_t[1,1] = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t},$$

3) We check if the chain has a reversible measure,

$$\mu(1)L[1,2] = \mu(2)L[2,1],$$

we find

$$\mu(1) = \frac{\beta}{\alpha+\beta} \qquad \text{and} \qquad \mu(2) = \frac{\alpha}{\alpha+\beta}.$$

□

Exercise 4 : A Markov triangle

We consider a Markov process X_t on a triangle, with vertices 1, 2, 3 going clockwise. In a small timestep dt, the process moves one step clockwise with probability $\alpha dt + O(dt^2)$, and counter clockwise with probability $\beta dt + O(dt^2)$, otherwise it stays put.

1) (i) Give the intensity matrix for this process, and give a graphic representation.

(ii) What is the probability starting from state 1, to hit state 2 before state 3 ?

(iii) We denote by T^3 the first time the process hits state 3. Compute $\mathbb{E}_1(T^3)$, which is the expectation starting from state 1 of the hitting time of state 3.

2) (i) What is the transition matrix of the skeleton (Y_k) of the process ?

(ii) What is the probability, for the skeleton, to hit state 2 before state 3, starting from state 1 ?

(iii) What is the expected number of steps in the skeleton to hit state 3 starting from state 1 ?

Answer :

1) (i)

$$\begin{pmatrix} -\alpha-\beta & \alpha & \beta \\ \beta & -\alpha-\beta & \alpha \\ \alpha & \beta & -\alpha-\beta \end{pmatrix}$$

(ii) The probability is $\alpha/(\alpha+\beta)$.

(iii) Either use the result from the lecture notes, or use the Markov property at the time of the first jump, to show that

$$\mathbb{E}_1(T^3) = \mathbb{E}_1(\tau_0) + \mathbb{P}_1(Y_1 = 2)\mathbb{E}_2(T^3) + \mathbb{P}_1(Y_1 = 3)\mathbb{E}_3(T^3) = \frac{1}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\mathbb{E}_2(T^3).$$

We apply once again the Markov property to the second term, to yield

$$\mathbb{E}_1(T^3) = \frac{1}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\left[\frac{1}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\mathbb{E}_1(T^3)\right],$$

so that

$$\mathbb{E}_1(T^3) = \frac{2\alpha+\beta}{\alpha^2+\beta^2+\alpha\beta}.$$

One can check that this coherent with the cases where $\alpha = 0$ ($T^3 \sim Exp(\beta)$) and $\beta = 0$ (T^3 is the sum of two independant $Exp(\alpha)$ variables).

2) (i)

$$\frac{1}{\alpha+\beta}\begin{pmatrix} 0 & \alpha & \beta \\ \beta & 0 & \alpha \\ \alpha & \beta & 0 \end{pmatrix}$$

(ii) It is $\alpha/(\alpha+\beta)$.

(iii) Each step in the continuous process takes an $Exp(\alpha+\beta)$ time, so that the number of steps needed in the ekeleton is $(\alpha+\beta)\mathbb{E}_1(T^3)$. □

Exercise 5 : A queue process

At the post office, clients arrive at a single counter and exit as they are taken care of. We assume that at a rate λ (i.e. at the ringing times of a rate λ Poisson clock), a random number of clients (independant from the poisson clock and from the other arrivals) arrives in the queue, according to a distribution $\mathbf{p} := (p_k)_{k\in\mathbb{N}}$ on $\mathbb{N}$. We assume that $p_0 = 0$. All clients are in a single line, and the first in the line is taken care of at a rate $\gamma > 0$.

1) Write down the intensity matrix for this process, and justify that the total number of clients N_t that have entered the post office before time t can be written

$$N_t = \sum_{k=1}^{Q_t} \xi_k,$$

where Q_t follows a Poisson distribution with a parameter that will be indicated, and the (ξ_k)'s are i.i.d. variables with distribution $\mathbf{p}$.

2) (i) Justify that the number M_t of people in the post office is an irreducible Markov process, and its unique communicating class is $\mathbb{N}$.

(ii) Show that M_t does not explode in finite time.

3) Assume that μ is a stationary probability distribution for M. We define $g(s) = \sum_k p_k s^k$ (resp $\psi(s) = \sum_k \mu_k s^k$) the probabillity generating function of $\mathbf{p}$. (resp. μ).

(i) Write down the equations satisfied by μ, and show that

$$\psi(s) = \frac{\gamma\mu_0(1-s)}{s\lambda(g(s)-1)+\gamma(1-s)}.$$

(ii) Let $m = \sum kp_k$ be the average of $\mathbf{p}$. Show that if there exists an invariant probability measure, we must have $m < \infty$ and $m\lambda/\gamma < 1$

(iii) Show that if $m\lambda/\gamma < 1$ there exits a unique invariant probability measure μ.

(iv) Assume $m\lambda/\gamma < 1$, compute $\mathbb{E}_\mu(X_t)$.

Answer : The intensity matrix is given by the transition rates

$$\ell_{k,k+\ell} = \begin{cases} \lambda p_\ell & \text{for } \ell \geq 1 \\ \gamma & \text{for } \ell = -1 \\ -\lambda-\gamma & \text{for } \ell = 0 \\ 0 & \text{otherwise} \end{cases}.$$

The number of occurences at which clients arrived follows a Poisson point process with rate λ, and at each occurence, an independent number of clients distributed as $\mathbf{p}$ arrives. Hence, choosing $Q_t \sim Poi(\lambda t)$, and ξ_k's distributed as i.i.d. $\mathbf{p}$, we obtain the formula.

1) (i) Choose an arbitrary k_0 such that $p_{k_0} > 0$. Since any positive integer ℓ

can be written

$$\ell = a_\ell k_0 - b_\ell,$$

for two positive integers a_ℓ, b_ℓ. Then, we have the path from any n to $n+\ell$ defined by the arrival of k_0 clients a_ℓ times, then the exit of b_ℓ successive clients. This path links n to $n+\ell$, and occurs at positive rate $(\lambda p_{k_0})^{a_\ell}\gamma^{b_{k_0}} > 0$. To go from $n+\ell$ to n we juste have to make ℓ clients leave, which occurs with rate $\gamma^\ell > 0$.

(ii) The process jumps at a total rate at most $\gamma + \lambda$ (except when there are no clients, where it jumps at rate λ), which is constant and therefore bounded, thus it does not explode.

2) (i) We apply the definition of a stationary distribution, we must have $\mathbb{E}_\mu(\mathscr{L}\mathbf{1}_{\{e\}}) = 0$ for any $k \in \mathbb{N}$. In particular,

$$\gamma\mu_1 = \lambda\mu_0, \qquad \text{and} \qquad (\gamma+\lambda)\mu_k = \gamma\mu_{k+1} + \lambda\sum_{\ell=1}^{k}\mu_{k-\ell}p_\ell \quad \text{for} \quad k \geq 1. \tag{1}$$

We now multiply this identity by s^k and sum over $k \geq 1$, to obtain that for any $s \neq 0$

$$(\gamma+\lambda)\sum_{k\geq 1} s^k\mu_k = \frac{\gamma}{s}\sum_{k\geq 1} s^{k+1}\mu_{k+1} + \lambda\sum_{1\leq \ell\leq k} s^k\mu_{k-\ell}p_\ell,$$

so that

$$(\gamma+\lambda)[\psi(s)-\mu_0] = \frac{\gamma}{s}[\psi(s)-\mu_0-s\mu_1] + \lambda g(s)\psi(s),$$

$$\left(\gamma+\lambda-\frac{\gamma}{s}-g(s)\right)\psi(s) = (\gamma+\lambda)\mu_0 - \frac{\gamma}{s}(\mu_0+s\mu_1) = \gamma\mu_0(1-1/s),$$

which proves the identity.

(ii) If there exists an invariant measure, ψ must be continuous in $s=1$, and we must have $\psi(1) = \sum_k \mu_k = 1$. We rewrite

$$s(g(s)-1) = \sum_{k\geq 1}(s^{k+1}-s)p_k = -(1-s)\sum_{k\geq 1}\sum_{j=1}^{k} s^j p_k,$$

so that

$$\psi(s) = \frac{\gamma\mu_0}{\gamma - \lambda\sum_{k\geq j\geq 1} s^j p_k} \underset{s\to 1}{\longrightarrow} \frac{\gamma\mu_0}{\gamma-\lambda m}.$$

Since γ, λ and μ_0 are non negative, in order to have $\psi(1) = 1$, we must have $m < \infty$ (otherwise $\psi(s) = 0 \neq 1$) and $\lambda m < \gamma$ (otherwise $\mu_0 < 0$).

(iii) If $\lambda m < \gamma$, we define $\mu_0 = (\gamma - \lambda m)/\gamma \in [0,1]$. We then determine μ_1 using the first identity in (1), and μ_k for $k \geq 2$ using the second part of (1). By construction, this yields an invariant measure, with total mass $1 = \psi(1)$. It is not hard to check taht by construction, $\mu_k > 0 \forall k$.

(iv) If $\lambda m < \gamma$, by stationarity

$$\mathbb{E}_\mu(M_t) = \mathbb{E}_\mu(M_0) = \sum_{k\geq 1} k\mu_k = \psi'(1),$$

which can be computed with either one of the forumlae for $\psi(s)$, since $g'(1) = m$, we obtain

$$\mathbb{E}_\mu(M_t) = \psi'(1) = \frac{\gamma\mu_0\lambda\sum_{k\geq j\geq 1} jp_k}{(\gamma-\lambda m)^2} = \frac{\lambda\sum_{k\geq 1}(k^2+k)p_k}{2(\gamma-\lambda m)}$$

□