Scaling limits of stochastic particle sytems

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Short curriculum

… before the PhD

- 2009 – 2013 : Student at **ENS Paris**
- 2011 – 2012 : "Probabilité et modèles aléatoires" **Master's degree at Paris VI** univ.
- 2013 – 2016 : **PhD at CMAP**, Polytechnique with Thierry Bodineau

… and after

- 2016 – 2018 : **Post-doc at IMPA**, Rio de Janeiro with Claudio Landim
- 2018 – 2020 : **Post-doc at Roma Tre**, Rome with Alessandro Giuliani
- 2020 – today : **ISFP at INRIA Lille**, in the PARADYSE project-team

PARADYSE Project team

PARADYSE : **PAR**ticles **A**nd **DY**namical **S**yst**E**ms

Three main research axis :

I) **Time asymptotics**: Stationary states, solitons, and (in)stability issues *(PDEs)*

II) Understanding **macroscopic** systems in terms of **microscopic** dynamics *(PDEs, Probability and Mathematical Physics)*

III) **Numerical methods**: analysis and simulations *(Numerics for PDEs and Probability)*

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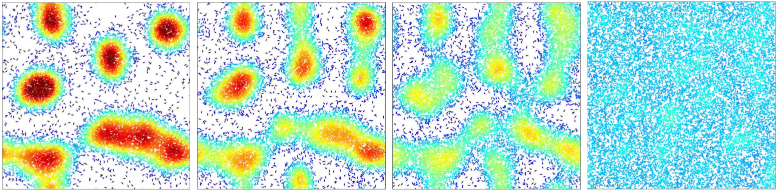
COMPLEX PHYSICAL SYSTEMS

- Many complex systems are composed of a **large number of microscopic or mesoscopic** components

Linking description levels

Problem : These microscopic components and their evolution is typically **not visible nor tractable** at our scale, but the macroscopic evolution of the system stems from its microscopic evolution.

- ▷ **Microscopic** level : e.g. particules suffer elastic collisions
  Problem : too many ($\sim 10^{23}$) degrees of freedom
- ▷ **Macroscopic** level : diffusion (heat equation).



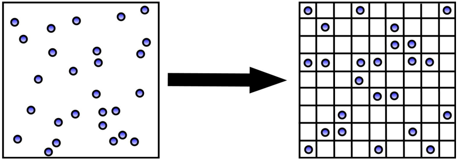
Objective : Model such **complex systems** to give a mathematical justification to their **macroscopic behavior**.

LATTICE GASES

QUESTION : how can one link **microscopic** and **macroscopic** scales ?

Need to **model** and **simplify** the physical system :

- ▷ Space and time **discretisation**. *Example* : exclusion processes, where each site of the lattice is either **occupied by a particle**, or **empty**.

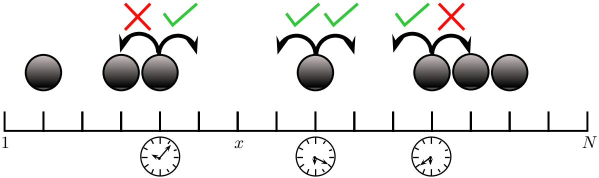


- ▷ **Deterministic** dynamics modeled by **stochastic** dynamics: particles jump at random times in random directions.

This type of models is **called stochastic lattice gases**

Symmetric Simple Exclusion Process (SSEP)

Configuration: each site x of the lattice $\{1, \ldots, N\}$ is either **empty** ($\eta_x = 0$) or **occupied** ($\eta_x = 1$).



Dynamics: each particle jumps at random times, either to the **left or to the right** w.p. $1/2$.

- If the target site is already occupied, the **jump is cancelled** (exclusion rule).

HYDRODYNAMIC LIMIT

We see the lattice as a discretization of the segment $[0,1]$, and define, for $u = x/N \in [0,1]$, the scaling limit $\rho(t,u)$ of $\eta_x(t)$,

$$\rho(t,u) = \lim_{N\to\infty} \frac{1}{2\sqrt{N}} \sum_{|y-x|\leq\sqrt{N}} \eta_y(t).$$

The **hydrodynamic limit** for the SSEP is given by the **heat equation**

$$\partial_t \rho = \partial_u^2 \rho$$

with initial condition

$$\rho(0,\cdot) = \rho_0$$

set by the initial microscopic configuration $\eta(0)$.

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SOME COMMENTS

- The terminology **hydrodynamic limit** stands for this procedure, of associating a *microscopic dynamics* with a *PDE* describing its macroscopic behavior.

- **Hydrodynamic limits** are the rough equivalent, for particle systems, of the **law of large numbers**, since it states that

$$\underbrace{\rho(t,u)}_{\textit{deterministic mean}} = \lim_{N\to\infty} \underbrace{\frac{1}{2\sqrt{N}} \sum_{|y-x|\leq\sqrt{N}} \eta_y(t)}_{\textit{average of random quantities}} .$$

- It can give access to physical phenomena such as **phase transitions** and **phase diagrams**, starting from a microscopic dynamics.

EXAMPLES OF APPLICATION

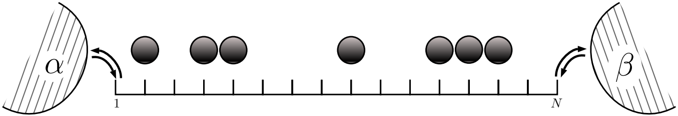
Exclusion processes like the SSEP can serve as **toy models** for various physical phenomena, e.g.:

- Heat transfer in **non-equilibrium systems** with boundary interactions.

- Liquid-solid phase transition with **Kinetically constrained** models.

Non-equilibrium models

To maintain a system out of equilibrium, we put it in contact **thermostats/particle reservoirs**. They are modeled for the SSEP by creation annihilation dynamics at the boundaries.



- At random time, site $x = 1$ is replaced by a $Bernoulli(\alpha)$, and site $x = N$ by $Bernoulli(\beta)$.

- The hydrodynamic behavior in the **bulk** $u \in (0,1)$ is still given by the heat equation $\partial_t \rho = \partial_u^2 \rho$.

- The **effect of the boundary dynamics** on the hydrodynamic limit depends on the **mean frequency** of the boundary updates.

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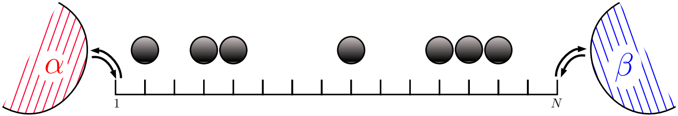

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SUBCRITICAL PHASE

Simulation by Hugo Dorfsman

- "Subcritical" frequency: **Neumann** boundary conditions

$$\partial_u \rho(t,0) = \partial_u \rho(t,1) = 0.$$

CRITICAL PHASE

Simulation by Hugo Dorfsman

- "Critical" frequency: **Robin** boundary conditions

$$\partial_u \rho(t,0) = \rho(t,0) - \alpha, \qquad \partial_u \rho(t,1) = \beta - \rho(t,1).$$

SUPERCRITICAL PHASE

Simulation by Hugo Dorfsman

- "Supercritical" frequency: **Dirichlet** boundary conditions

$$\rho(t,0) = \alpha, \qquad \rho(t,1) = \beta.$$

A few related works

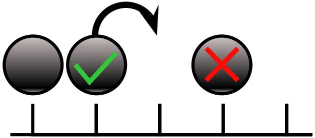
- *Hydrodynamic limit of boundary driven exclusion processes with nonreversible boundary dynamics*, E', J. Stat. Phys., 172, pp 1327–1357 (2018).

- *Hydrodynamics for SSEP with non-reversible slow boundary dynamics*, E', Gonçalves, and Nahum, J. Stat. Phys., 181, pp 1433–1469 (2020) and ALEA v.17, pp 791-823 (2020).

- *Steady state large deviations for one-dimensional, symmetric exclusion processes in weak contact with reservoirs*, Bouley, E', and Landim, Arxiv Preprint 2107.06606 (2021).

KINETICALLY CONSTRAINED LATTICE GASES

QUESTION: what happens when a kinetic constraint is added to the system ? example : **facilitated exclusion process** (FEP):



- A particle cannot jump to an occupied site (*exclusion rule*)
- A particle cannot jump without occupied neighbor (*kinetic constraint*)

Because of the kinetic constraint, two typical particle behavior:

- At **low density** , they quickly freezes
- At **high density** ($\rho > 1/2$), they roughly behave like in the SSEP
- The **critical density** is when every other site is occupied, i.e. $1/2$

Kinetically constrained lattice gases

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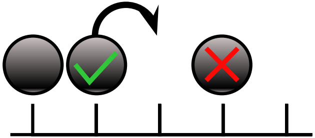

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Hydrodynamic limit for the FEP

The two microscopic phases translate as macroscopic phases at the hydrodynamic limit:

- in the **subcritical** phase ($\rho < 1/2$), the system is **frozen**
- in the **supercritical** phase ($\rho < 1/2$), the system is **diffusive**

The hydrodynamic limit is a **Stefan problem**, where the diffusive phase invades the frozen phase through critical interfaces:

$$\partial_t \rho = \partial_u^2 \left\{ \frac{2\rho - 1}{\rho} \mathbf{1}_{\{\rho \geq 1/2\}} \right\}$$

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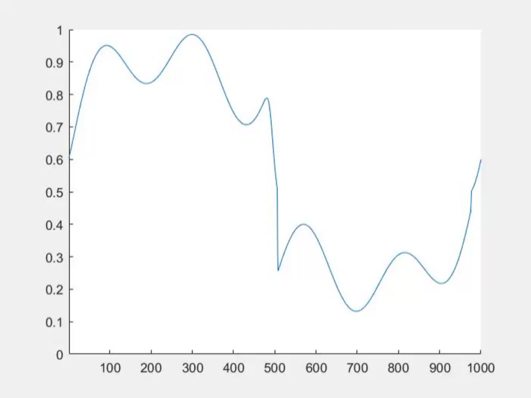

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A few related works

- *Hydrodynamic limit for a facilitated exclusion process*, Blondel, E', Sasada and Simon, Annales de l'IHP - Prob. et Stat., v.56, Iss.1, pp 667–714, (2020).
- *Stefan problem for a non-ergodic facilitated exclusion process*, Blondel, E' and Simon, Prob. and Math. Phys., v.2, Iss.1, pp 127-178 (2021).
- *Mapping hydrodynamics for the facilitated exclusion and zero-range processes*, E', Simon and Zhao, Arxiv Preprint 2202.04469 (2022).

Take-away message

- Stochastic lattice gases provide useful **mathematically tractable models** for a wide variety of physical behavior.
- Scaling limits, and hydrodynamic limits in particular, can be used to understand the **macroscopic behavior** of lattice gases.
- In some cases, this allows to get access, from the microscopic dynamics, to **phase diagrams** and other physically relevant information.