Using duality to derive hydrostatic and hydrodynamic limits

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Plan of the talk

General setting

- Consider $\Lambda_N = \{1, \dots, N\}$ and the set of configurations $\Omega_N = \{0,1\}^{\Lambda_N}$.
- For any configuration η, we define an infinitesimal generator $\mathscr{L}_N$ acting on functions of η.
- Assume that $\mathscr{L}_N$ admits a unique stationary measure μ^N, i.e. such that for any function $f : \Omega_N \to \mathbb{R}$,

$$\mathbb{E}_{\mu^N}(\mathscr{L}_N f) = 0.$$

- We denote by ν_α^N the product measure on Ω_N with density α.

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Dirichlet form

Fix $\alpha \in]0,1[$, we split

$$\mathscr{L}_N = \mathscr{L}_N^s + \mathscr{L}_N^a,$$

resp. the self adjoint and anti-self adjoint parts of the generator in $L^2(\nu_\alpha^N)$.
We can then define the Dirichlet form

$$D_N(f) = \mathbb{E}_\alpha\left(f(-\mathscr{L}_N^s)f\right),$$

which is positive and convex.

Entropy method

Consider a Markov process $\eta(t)$, started from a measure μ_0^N and driven by $\mathscr{L}_N$. We denote μ_s^N the distribution of $\eta(s)$ and $f_s^N = d\mu_s^N / d\nu_\alpha^N$.

For the entropy

$$H(f) = \mathbb{E}_{\nu_\alpha^N}(f \log f),$$

we can usually write

$$H(f_t^N) + \int_0^t ds D_N(f_s^N) \leq CN,$$

which is the basis for the entropy method.

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The dynamics

One-dimensional process with a three parts dynamics

- **Bulk** : each pair of sites $k, k+1$ is exchanged at rate 1.
- **Right boundary** : in contact with a reservoir at equilibrium, at density $\beta \in [0,1]$. The last site is filled at rate β and emptied at rate $1-\beta$.
- **Left boundary** : the two first sites are in contact with two different reservoirs at different densities α_1 and α_2.

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Generator for the model

The generator is given by

$$\mathscr{L}_N = \mathscr{L}_N^l + \mathscr{L}_N^b + \mathscr{L}_N^r,$$

where

$$\mathscr{L}_N^b f = \sum_{k=1}^{N-1} \big[f(\eta^{x,x+1}) - f(\eta) \big].$$

$$\mathscr{L}_N^b f = (\beta(1-\eta_N) + (1-\beta)\eta_N) \big[f(\eta^N) - f(\eta) \big].$$

$$\mathscr{L}_N^l f = (\alpha_1(1-\eta_1) + (1-\alpha_1)\eta_1) \big[f(\eta^1) - f(\eta) \big] + (\alpha_2(1-\eta_2) + (1-\alpha_2)\eta_2) \big[f(\eta^2) - f(\eta) \big].$$

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We denote μ^N the unique stationnary measure w.r.t. $\mathscr{L}_N$.

Hydrostatic limit

For any positive δ, and any smooth function $H : [0,1] \to \mathbb{R}$, we have

$$\limsup_{N\to\infty} \mu^N \left(\left| \frac{1}{N} \sum_{k\in\Lambda_N} H(k/N)\eta_k - \int_0^1 H(u)\rho(u)du \right| > \delta \right) \to 0,$$

where ρ is the unique weak solution to

$$\begin{cases} \Delta\rho = 0 \\ \rho(0) = (\alpha_1 + 2\alpha_2)/3 \\ \rho(1) = \beta \end{cases} .$$

Density and correlations

We estimate the left density and the correlations using a coupling with random walks by the Feynman kac formula. We let

$$\rho_N(k) = \mathbb{E}_{\mu^N}(\eta_k).$$

→ Since μ^N is a stationnary measure, for any function f of the configuration,

$$\mathbb{E}_{\mu^N}(\mathscr{L}_N f) = 0.$$

This yields in particular

$$\begin{cases} (\Delta_N \rho_N)(k) := \rho_N(k+1) + \rho_N(k-1) - 2\rho_N(k) = 0 \\ \rho_N(2) + \alpha_1 - 2\rho_N(1) = 0 \\ \rho_N(1) + \rho_N(3) + \alpha_2 - 3\rho_N(2) = 0 \\ \rho_N(N-1) + \beta - 2\rho_N(N) = 0 \end{cases}.$$

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Random walk and cemetery states

Define three cemetery states $\eth_1$, $\eth_2$ and $\eth_N$, and let (X_t) be a random walk on $\Lambda_N \cup \{\eth_1, \eth_2, \eth_N\}$, such that

- When $X = k \in \Lambda_N$, X jumps to any neighbor at rate 1.
- When $X = 1$, (resp. $k = 2$), X also jumps at rate 1 to $\eth_1$ (resp. $\eth_2$).
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Coupling

Let

$$\rho_N(\mathfrak{d}_1) = \alpha_1, \qquad \rho_N(\mathfrak{d}_2) = \alpha_2 \quad \text{and} \quad \rho_N(\mathfrak{d}_N) = \beta,$$

Then, we can write with Feynman-Kac's formula

$$\rho_N(k) = \mathbb{E}_k(\rho_N(X_\tau)) := \mathbb{E}(\rho_N(X_\tau) \mid X_0 = k)$$

where

$$\tau = \inf\{s \geq 0, \quad X_s \in \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_N\}\}.$$

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Computing ρ_N

- Since $(\Delta_N \rho_N)(k) = 0$, $\forall\ 3 \leq k \leq N-1$, ρ_N is affine in $\{3, \ldots, N-1\}$, and

$$\rho_N(k) = \frac{N-k}{N-2}\rho_N(2) + \frac{k-2}{N-2}\rho_N(N).$$

- $\rho_N(N) = \mathbb{E}_N(\rho_N(X_\tau)) = \beta + O(1/N)$
- $\rho_N(2) = \mathbb{E}_2(\rho_N(X_\tau)) = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 + O(1/N)$

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Correlations (2)

We do the same with the correlations

$$\varphi_N(k,l) = \mathbb{E}_{\mu^N}(\eta_k \eta_l) - \rho_N(k)\rho_N(l), \quad 1 \le k < l \le N.$$

To compute $\varphi_N(k,l)$, we now consider two random walks X^1 and X^2 on

$$\bar{\Lambda}_N = \Lambda_N \cup \{\partial_1, \partial_2, \partial_N\}.$$

$X = (X^1, X^2)$ is a random walk on

$$\left\{(x_1, x_2) \in \bar{\Lambda}_N^2, \quad x_1 \neq x_2\right\}.$$

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Correlations (1)

We can write for the correlations

$$\varphi_N(k,l) = \mathbb{E}_{(k,l)}(\varphi_N(\boldsymbol{X}_\tau)) + \mathbb{E}_{k,l}\left(\int_{t=0}^{\tau} ds\, \mathbf{1}_{|X_s^1 - X_s^2|=1}(\rho_N(X_s^2) - \rho_N(X_s^1))^2\right),$$

where

$$\tau = \inf\{s \geq 0, \quad X_s^1 \text{ or } X_s^2 \in \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_N\}\}.$$

We obtain

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Proof of the hydrostatic limit

We estimate

$$\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\sum_{k\in\Lambda_N}H(k/N)\eta_k-\int_0^1 H(u)\rho(u)du\right|\right)$$

$$\leq \mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\sum_{k\in\Lambda_N}H(k/N)(\eta_k-\rho_N(k))\right|\right)$$

$$+\mathbb{E}_{\mu^N}\left(\left|\frac{1}{N}\sum_{k\in\Lambda_N}H(k/N)\rho_N(k)-\int_0^1 H(u)\rho(u)du\right|\right)$$

The second term is controlled by our estimation of the density, the first one is controlled by the bound on the correlations.

Hydrodynamic limit

Going from the hydrostatic limit to the hydrodynamic limit adds technical difficulties.

- to estimate $\rho_N(t,k)$ and $\varphi_N(t,k,l)$, the random walks X and $\boldsymbol{X}$ are launched at time 0, back in time.
- We can write in particular

$$\rho_N(t,k) = \mathbb{E}_k(\rho_N(X_{t-\tau\wedge t}))$$

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Left boundary condition : autonomous equation

This method can be generalized to a boundary of size p, as long as for any $1 \leq k \leq p$, we can write

$$\mathscr{L}_N \eta_k = r_k(\alpha_k - \eta_k) + \sum_{l=1}^{p} q_{k,l}(\eta_l - \eta_k).$$

This is, however, quite a restrictive condition : under this assumption, the only elements allowed in te border dynamics are

- Reservoirs : Site k is updated at rate r_k by a equilibrium reservoir at density α_k.
- Stirring : Sites k,l are exchanged at rate $s_{k,l}$.
- Copy : Site k "copies" site l at rate $c_{k,l}$.

general rates for the left boundary

We now want to generalize the method above, and let

$$\mathcal{L}_N^l f = c(\eta_1, \ldots, \eta_p)\left[f(\eta^1) - f(\eta)\right].$$

Let

$$A = \min\{c(0, \eta_2, \ldots, \eta_p)\} \text{ and } B = \min\{c(1, \eta_2, \ldots, \eta_p)\},$$

we assume that

$$\max\{c(0, \eta_2, \ldots, \eta_p)\} - A \leq \frac{A+B}{(p-1)2^{p-1}}$$

and

$$\max\{c(1, \eta_2, \ldots, \eta_p)\} - B \leq \frac{A+B}{(p-1)2^{p-1}}$$

Then, the left generator can be rewritten

$$\mathscr{L}_N^l f = \lambda^+(\eta_1,\ldots,\eta_p)\big[f(C^1\eta)-f(\eta)\big]+\lambda^-(\eta_1,\ldots,\eta_p)\big[f(A^1\eta)-f(\eta)\big] + A\big[f(C^1\eta)-f(\eta)\big]+B\big[f(A^1\eta)-f(\eta)\big],$$

where

$$\lambda^+(\eta_1,\ldots,\eta_p)=(1-\eta_1)(c(0,\eta_2,\ldots,\eta_p)-A)$$

and

$$\lambda^-(\eta_1,\ldots,\eta_p)=\eta_1(c(1,\eta_2,\ldots,\eta_p)-B)$$

We construct graphically the process η.

Thanks for your attention !