Detailed summary of the thesis

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Non-asymptotic estimates of invariant measure and regularisation by a degenerate noise for a chain of Ordinary Differential Equations

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Short abstract

In my thesis, two topics are studied. The first one deals with invariant measures of stochastic processes and their approximations. In different contexts, some non-asymptotic estimations of the error by Gaussian concentration inequalities are stated. This yields some sharp and non-asymptotic controls of the confidence intervals.

The second part is about the regularization by the noise. First of all, it is used through Schauder estimates. We establish these controls for a Kolmogorov Partial Differential Equation (PDE) associated with a degenerate Stochastic Differential Equation (SDE). We show that the solution has a *parabolic* regularity gain corresponding to the self-similarity of the considered noise.

Next, the strong uniqueness of the solution is proved for the previously considered SDE. This stochastic equation can be regarded as an Ordinary Differential Equation (ODE) with a noise. The self-similar noise restores the uniqueness of the ODE solution in a Hölder framework.

I) Non-asymptotic concentration inequalities (Part II of the thesis)

Foreword

The first part of this summary is divided into 4 sections. Section 1 shows the motivations and the difficulties to approximate the invariant measure of a diffusion process. In Section 2, corresponding to the Chapter 3 in my thesis, we set the considered approximation scheme, I give the main concentration inequalities obtained in this chapter, and the sketch of the proofs. In this section, I state as well some pointwise controls of the solution of the associated Poisson equation. In Section 3, relying on Chapter 4 of my thesis, there is a sharp concentration result. Section 4, dealing with Chapter 5 of my thesis, I briefly explain how to extend the method developed in the Brownian case to a certain kind of Lévy process.

1 Estimation of invariant measures

The laws of physics and biochemistry lead to consider some stochastic models whose the asymptotic behaviour is crucial. The equations from Hamiltonian mechanics (such as molecular dynamics van der Waals forces, cf. [LRS10]), fluid mechanics with Navier-Stokes equations (e.g. [HM06] in an infinite dimensional case), the models of neural circuit in neuroscience (cf. [FM14]) are some examples.

In a numerical point of view, calculating the average value of a large number of samples of the processes, up to a time big enough, has cost computation very high. We then aim to take advantage of some ergodic properties of the considered process to simulate only one trajectory. For practical purposes, we establish some non-asymptotic ergodic estimates.

Let us consider the stochastic process $(\mathbf{X}_t)_{t\geqslant 0}$ given by the following dynamic:

$$d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t,\ t > 0, \tag{1.1}$$

where $(W_t)_{t\geqslant 0}$ is a Brownian motion of dimension $r \in \mathbb{N}^*$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$ (potentially degenerate) are Lipschitz continuous functions. We suppose that the process $(\mathbf{X}_t)_{t\geqslant 0}$ is ergodic and has a unique invariant distribution ν. This distribution is said to invariant if for any function f smooth enough and any $t \geqslant 0$,

$$\int P_t f(x)\nu(dx) = \nu(f) =: \int f(x)\nu(dx),$$

where is the associated semi-group defined by $P_t f(x) := \mathbb{E}[f(\mathbf{X}_t)|\mathbf{X}_0 = x]$. The stationary process with ν as invariant distribution is ergodic if for any bounded continuous function f one has almost surely (a.s.)

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t f(X_s)ds =: \lim_{t\to+\infty} \nu_t(f) = \nu(f) = \int f(x)\nu(dx). \tag{1.2}$$

The convergence rate towards the invariant distribution is given by Central Limit Theorem (CLT). For the diffusion (1.1), Bhattacharya [Bha82], under some irreducibility assumptions (non-degenerate diffusions), established the corresponding CLT: for any function f with polynomial growth,

$$\sqrt{t}\big(\nu_t(f-\nu(f))\big) \xrightarrow[t\to+\infty]{\text{(loi)}} \mathcal{N}\Big(0, \int_{\mathbb{R}^d} |\sigma^*\nabla\varphi(x)|^2\nu(dx)\Big), \tag{1.3}$$

with φ the solution of the Poisson equation

$$\mathcal{A}\varphi = f - \nu(f). \tag{1.4}$$

In practice, we have to use a discretization procedure to approximate the invariant distribution. A standard approach is to use the Euler scheme with constant time step $\gamma > 0$ associated with (1.1):

$$X_{n+1}^{\gamma} = X_n^{\gamma} + \gamma b(X_n^{\gamma}) + \sqrt{\gamma}\sigma(X_n^{\gamma})U_{n+1},$$

where $(U_n)_{n\geqslant 1}$ is a sequence of random variables $\mathbb{R}^r$ valued independent and identically distributed (i.i.d.). This sequence corresponds to the Brownian increments, see [TT90].

However, the ergodic theorem yields that for any bounded continuous function f $\nu_n^{\gamma}(f) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \nu^{\gamma}(f)$, where ν^{γ} is the invariant distribution of the scheme and not the one of the diffusion (1.1). Therefore, this scheme is asymptotically biased which corresponds to the discretization error. To avoid this problem, we used an algorithm with decreasing time steps.

2 A first concentration result (Chapter 3)

Under some Lyapunov assumptions, we establish that the approximation error of the invariant distribution by the Lamberton Pagès scheme satisfies a Gaussian concentration inequality. This result strongly relies on the regularity of the solution of the Poisson equation associated with the considered stochastic dynamic. Indeed, we develop the scheme thanks to a distretized version of Itô's lemma.

2.1 Scheme with decreasing step

The first chapter of the thesis comes from a collaboration with S. Menozzi (LaMME, UEVE) and G. Pagès (LPSM, Paris 6) which is to appear in *Annales de l'Institut Henri Poincaré*, [HMP19]. We establish non-asymptotic Gaussian concentration inequalities associated with a decreasing time step scheme. This algorithm was introduced by [PS94], [BHW97]. Next Lamberton and Pagès [LP02] established the ergodic theorem and the unbiased CLT associated with this scheme. In other words, this approach allows to compute the invariant distribution asymptotically without bias error. We set for any $n \geqslant 0$:

$$X_{n+1} = X_n + \gamma_{n+1}b(X_n) + \sqrt{\gamma_{n+1}}\sigma(X_n)U_{n+1}. \tag{2.1}$$

The empirical measure is defined as following: for any Borelian set A,

$$\nu_n(A) := \nu_n(\omega, A) := \frac{\sum_{k=1}^n \gamma_k \delta_{X_{k-1}(\omega)}(A)}{\Gamma_n},\ \Gamma_n := \sum_{k=1}^n \gamma_k. \tag{2.2}$$

Let us remark that Γ_n is the discrete counterpart of the time t considered in (1.2). For study of the long time behaviour, the sequence of time steps $(\gamma_k)_{k\geqslant 1}$ is chosen such that $\Gamma_n \underset{n}{\to} +\infty$.

Under some suitable Lyapunov assumptions, Lamberton and Pagès establish the ergodic theorem associated to the algorithm (2.1), i.e. a discretized version (1.2): for any bounded continuous function f,

$$\nu_n(f) \xrightarrow[n\to+\infty]{a.s.} \nu(f) = \int_{\mathbb{R}^d} f(x)\nu(dx). \tag{2.3}$$

Let us suppose now that the following Lyapunov assumptions are in force.

($\mathcal{L}_\mathbf{V}$) There is a function $V : \mathbb{R}^d \longrightarrow (0, +\infty[$, satisfying the following conditions:

i) Regularity-Coercivity. V is $\mathcal{C}^2$ function such that $\|D^2V\|_\infty < +\infty$, $\lim_{|x|\to\infty} V(x) = +\infty$.

ii) Growth control. There is $C_V \in (0, +\infty)$ such that for any $x \in \mathbb{R}^d$:

$$|\nabla V(x)|^2 + |b(x)|^2 \leqslant C_V V(x).$$

iii) Stability. There are $\alpha_V > 0$, $\beta_V \in \mathbb{R}^+$ such that for any $x \in \mathbb{R}^d$,

$$\mathcal{A}V(x) \leqslant -\alpha_V V(x) + \beta_V.$$

These conditions yield that V is sub-quadratic and b sub-linear, the process $(X_t)_{t\geqslant 0}$ then "looks like" an Ornstein-Uhlenbeck process.

From [LP02], if there is a unique invariant distribution of the diffusion (1.1) and if $\lim_n \frac{\sum_{k=1}^n \gamma_k^2}{\sqrt{\Gamma_n}} = 0$, then for any function φ:

$$\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \xrightarrow[n\to+\infty]{\text{(loi)}} \mathcal{N}\left(0, \int_{\mathbb{R}^d} |\sigma^*\nabla\varphi|^2 d\nu\right). \tag{2.4}$$

For a polynomial time step, if there is $\theta \in [0, 1]$ such that $\gamma_k \asymp k^{-\theta}$, the condition $\lim_n \frac{\sum_{k=1}^n \gamma_k^2}{\sqrt{\Gamma_n}} = 0$ is equivalent to $\theta \in (1/3, 1]$. Therefore, under this step constraint, there is no discretization error for the convergence rate of the scheme.

For the critical case, $\theta = 1/3$, the convergence rate increases. The CLT is still available but a bias appears due to a time step too big.

For $\theta < 1/3$, there is no CLT. The discretization error "hides", somehow, the CLT. Only a convergence in probability of the renormalized error $\frac{\Gamma_n}{\sum_{k=1}^n \gamma_k^2}\big(\nu(f) - \nu(f)\big)$ holds.

2.2 Concentration inequalities

From now on, we suppose that the innovation $(U_k)_{k\geqslant 1}$ of the scheme (2.1) satisfies the Gaussian concentration property:

(GC) The random variable U has the same three first moments as the standard normal law $\mathcal{N}(0, I_r)$, and for any $1-$Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}$, and any $\lambda > 0$:

$$\mathbb{E}\big[\exp(\lambda g(U))\big] \leqslant \exp\left(\lambda \mathbb{E}\left[g(U)\right] + \frac{\lambda^2}{2}\right).$$

This property means that the tail of the law of U is sub-Gaussian. In particular, this holds for the standard normal law, the Rademacher's law, and for any probability law satisfying log-Sobolev inequalities (also called Gross inequalities, cf. [Roy07]).

We obtain under these assumptions, for ν_n defined in (2.2), for all $n > 0$ and $a = a(n) > 0$:

$$\mathbb{P}[\sqrt{\Gamma_n}|\nu_n(f) - \nu(f)| \geqslant a] \leqslant C_n \exp\Big(- c_n \frac{a^2}{2\|\sigma\|_\infty^2 \|\nabla\varphi\|_\infty^2}\Big), \tag{2.5}$$

with $C_n, c_n > 0$ such that $\lim_n C_n = \lim_n c_n = 1$. This result is available for any test function f lying in an appropriate function class (smooth enough, here $C^{1,\beta}$: the derivative is bounded and β-Hölder continuous) such that $f - \nu(f)$ is coboundary of the infinitesimal generator. In other words, the Poisson equation

$$f - \nu(f) = \mathcal{A}\varphi, \tag{2.6}$$

has to be well-posed, with the solution φ smooth enough, $\mathcal{A}$ the infinitesimal associated with (1.1).

As an important by-product of this concentration result (2.5), we obtain the following estimate:

$$\mathbb{P}\Big[\nu(f) \in \big[\nu_n(f) - \frac{a}{\sqrt{\Gamma_n}}, \nu_n(f) + \frac{a}{\sqrt{\Gamma_n}}\big]\Big] \geqslant 1 - 2C_n \exp\big(- c_n \frac{a^2}{2\|\sigma\|_\infty^2 \|\nabla\varphi\|_\infty^2)}\big).$$

The concentration constant $\|\sigma\|_\infty^2 \|\nabla\varphi\|_\infty^2$ is not sharp, in the sense that it does not match the variance limit of the CLT (2.4) which is $\nu(|\sigma^*\nabla\varphi|^2)$, called *carré du champ*. We have readily the inequality: $\|\sigma\|_\infty^2 \|\nabla\varphi\|_\infty^2 \geqslant \nu(|\sigma^*\nabla\varphi|^2)$, which can highly change the control of the convergence rate of the algorithm.

We show that these bounds can be improved when the square Frobenius norm (or any norm upper-bounding the operator norm) of the diffusion coefficient σ lies in the same coboundary class as f. We obtain for a large class of deviation ($a = o(\sqrt{\Gamma_n})$) a sharper inequality than (2.5)

$$\mathbb{P}[\sqrt{\Gamma_n}|\nu_n(f) - \nu(f)| \geqslant a] \leqslant C_n \exp\Big(- c_n \frac{a^2}{2\nu(\|\sigma\|^2)\nabla\varphi\|_\infty^2}\Big).$$

We also establish non-asymptotic concentration inequalities for the almost sure Central Limit Theorem.

In the previous concentration inequality, the term $\nu(\|\sigma\|^2)\|\nabla\varphi\|_\infty^2$ is an approximation of *carré du champ* which depends on the invariant distribution ν that we want to estimate. To bypass this difficulty, after a suitable renormalization we establish a Slutsky like result:

$$\mathbb{P}[\sqrt{\Gamma_n}\frac{|\nu_n(f) - \nu(f)|}{\sqrt{\nu_n(\|\sigma\|^2)}} \geqslant a] \leqslant C_n \exp\Big(- c_n \frac{a^2}{2\|\nabla\varphi\|_\infty^2}\Big).$$

We also improve these first results to be able to consider Lipschitz continuous test function f. This is Lipschitz framework is usual for functional inequalities (optimal transport, Talagrand inequalities...). This context leads to an addition constraint on the time step, $\theta > \frac{1}{2}$, which decreases the convergence rate.

2.3 Sketch of the proof for the inequality (2.5).

The analysis is biased on the martingale method called Azuma's approach, cf. [Azu67]. The starting point is to use Bienaymé-Tchebyshev exponential inequality for any $\lambda > 0$:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geqslant a\big] \leqslant \exp(-\frac{\lambda a}{\sqrt{\Gamma_n}})\mathbb{E}[\exp(\lambda\nu_n(\mathcal{A}\varphi))]. \tag{2.7}$$

Next, we use a Taylor expansion between $\varphi(X_k)$ and $\varphi(X_{k-1})$ to make appear the infinitesimal generator associated with the dynamic (1.1). We obtain, somehow, a discretization form of the Itô formula:

$$\varphi(X_k) - \varphi(X_{k-1}) = \gamma_k\mathcal{A}\varphi(X_{k-1}) + \Delta_k(X_{k-1}, U_k) + R_{k,k-1}, \tag{2.8}$$

where $R_{k,k-1}$ is a remainder, $\Delta_k(X_{k-1}, U_k)$ is a martingale increment and where the mapping $u \mapsto \Delta_k(X_{k-1}, u)$ is Lipschitz continuous. Indeed, in the equality (2.8), in the right-hand side, only $\psi_k(X_{k-1}, U_k)$ depends on de U_k while in the left-hand side $\varphi(X_k)$ is Lipschitz continuous for the variable U_k, where

$$[\Delta_k(X_{k-1}, \cdot)]_1 \leqslant \sqrt{\gamma_k}\|\sigma(X_{k-1})\|\|\nabla\varphi\|_\infty \leqslant \sqrt{\gamma_k}\|\sigma\|_\infty\|\nabla\varphi\|_\infty, \text{ a.s.} \tag{2.9}$$

We sum the Taylor expansion (2.8):

$$\Gamma_n\nu_n(\mathcal{A}\varphi) = -M_n - R_n,$$

where $R_n = \varphi(X_0) - \varphi(X_n) + \sum_{k=1}^n R_{k,k-1}$ is a remainder term and $M_n = \sum_{k=1}^n \Delta_k(X_{k-1}, U_k)$ is a martingale term martingale which implies the Gaussian concentration. The analysis of the remainder contribution R_n is technical, the main argument is biased on the exponential integrability of the Lyapunov function V.

From (2.7) and Hölder's inequality for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geqslant a\big] \leqslant \exp(-\frac{\lambda a}{\sqrt{\Gamma_n}})\mathbb{E}[\exp(-\frac{\lambda q M_n}{\Gamma_n})]^{1/q}\mathbb{E}[\exp(\frac{\lambda p R_n}{\Gamma_n})]^{1/p}. \tag{2.10}$$

We show that we can choose $p = p(n) \to_n +\infty$ such that $\mathbb{E}[\exp(\frac{\lambda p R_n}{\Gamma_n})]^{1/p} = \mathscr{R}_n \to_n 1$. By conditional expectation property $\mathbb{E}[e^{-\frac{\lambda q M_n}{\Gamma_n}}] = \mathbb{E}\big[e^{-\frac{\lambda q M_{n-1}}{\Gamma_n}}\mathbb{E}[e^{\frac{\lambda q \Delta_n(X_{n-1}, U_n)}{\Gamma_n}}|\mathcal{F}_{n-1}]\big]$, by inequality (2.10) and using **(GC)**, one has:

$$\begin{aligned}\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \geqslant a\big] &\leqslant \mathscr{R}_n \exp(-\frac{\lambda a}{\sqrt{\Gamma_n}})\mathbb{E}[\exp(-\frac{\lambda q M_n}{\Gamma_n})]^{1/q} \\ &\overset{\textbf{(GC)}}{\leqslant} \mathscr{R}_n \exp(-\frac{\lambda a}{\sqrt{\Gamma_n}} + \frac{\lambda^2 q\gamma_n\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}{2\Gamma_n^2})\mathbb{E}\big[\exp(-\frac{\lambda q M_{n-1}}{\Gamma_n})\big]^{1/q}.\end{aligned} \tag{2.11}$$

We iterate this argument to deduce:

$$\mathbb{P}\big[\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geqslant a\big] \leqslant \exp(-\frac{\lambda a}{\sqrt{\Gamma_n}} + \frac{q\lambda^2\|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2}{2\Gamma_n}).$$

An optimization with respect to λ leads to the desired result.

2.4 Study of the Poisson equation

These concentration results require a gradient control of solution of the Poisson equation associated with the SDE. We obtain such an inequality in an elliptic and a mild confluence framework, and as well in a smooth and strong confluence framework. The considered confluence assumption write: $(\mathbf{D}_\alpha^p)$ There are $\alpha > 0$ and $p \in (1, 2]$ such that for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$

$$\left\langle \frac{Db(x) + Db(x)^*}{2}\xi, \xi \right\rangle + \frac{1}{2}\sum_{j=1}^{r}\Big((p-2)\frac{|\langle D\sigma_{\cdot j}(x)\xi, \xi\rangle|^2}{|\xi|^2} + |D\sigma_{\cdot j}\xi|^2\Big) \leqslant -\alpha|\xi|^2.$$

This technical condition means that the fluctuations of the derivative of b have to make the process coming back the origin more than the variations of the diffusion matrix σ.

When $(\mathcal{L}_{\mathbf{V}})$ and $(\mathbf{D}_\alpha^p)$ are in force, we show that if:

- [*U.E., mild confluence*] σ is U.E., $\|D\sigma\|_\infty^2 \leqslant \frac{2\alpha}{2(1+\beta)-p}$, $b, \sigma, f \in C^{1,\beta}$, and a structure assumption on σ

or

- [*Regularity, strong confluence*] $\|D\sigma\|_\infty^2 \leqslant \frac{2\alpha}{2(3+\beta)-p}$ and b, σ, f being $C^{3,\beta}$

then there is a unique invariant distribution associated with the dynamic (1.1). There is a unique solution $\varphi \in \mathcal{C}^{3,\beta}(\mathbb{R}^d, \mathbb{R})$, $\beta \in (0, 1)$, centered in ν of the Poisson equation

$$\forall x \in \mathbb{R}^d,\ \mathcal{A}\varphi(x) = f(x) - \nu(f), \tag{2.12}$$

which satisfies:

$$\|\nabla\varphi\|_\infty \leqslant \frac{[f]_1}{\alpha}.$$

To establish these properties, we use the Feynman-Kac probabilistic representation of the solution of (2.12),

$$\varphi(x) = \mathbb{E}_x\Big[\int_0^{+\infty} f(\mathbf{X}_t) - \nu(f)dt\Big], \tag{2.13}$$

next we differentiate the flow “à la Kunita” [Kun97]. In the U.E. case, we use the Schauder estimates of Krylov and Priola [KP10]. In the regular framework, we iteratively differentiate the flow and the confluence assumption allows us to integrate in time the derivatives of (2.13).

3 Sharp concentration inequality (Chapter 4)

We get the optimal concentration inequality of the approximation error of the invariant distribution by Lamberton Pagès scheme. This result is obtained by considering a new Poisson equation, where naturally appears that the asymptotic limit in the CLT is coboundary.

This chapter comes from a work alone [Hon19], which is accepted for publication in *Stochastic Processes and their Applications*.

If there is a function $\vartheta \in \mathcal{C}^{3,\beta}$ such that $\mathcal{A}\vartheta = |\sigma^*\nabla\varphi|^2 - \nu(|\sigma^*\nabla\varphi|^2)$, then, for any $\theta \in (\frac{1}{2+\beta}, 1]$, $n \geqslant 1$, $a = a(n) > 0$ satisfying $a/\sqrt{\Gamma_n} \to 0$ (*Gaussian deviation*):

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(f) - \nu(f)| \geqslant a\big] \leqslant 2\,C_n \exp\big(-c_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big), \tag{3.1}$$

with φ smooth enough such that $\mathcal{A}\varphi = f - \nu(f)$ and $C_n, c_n > 0$ going to 1 when $n \to \infty$. These considered concentration regimes, $a = o(\sqrt{\Gamma_n})$, include the confidence intervals case ($\Gamma_n \to_n +\infty$). Hence the concentration constant corresponds to the asymptotic variance in the CLT (2.4). We extend this result to a Lipschitz continuous source when the diffusion is non-degenerate.

The main idea of the proof consists in improving the concentration inequality used in (2.11). To do that, we have to control the Lipschitz modulus appearing in (2.9) more accurately . Inequality (3.1) yields some optimal controls of confidence intervals:

$$\mathbb{P}\Big[\nu(f) \in \big[\nu_n(f) - \frac{a}{\sqrt{\Gamma_n}}, \nu_n(f) + \frac{a}{\sqrt{\Gamma_n}}\big]\Big] \geqslant 1 - 2C_n \exp\big(-c_n \frac{a^2}{2\nu(|\sigma^*\nabla\varphi|^2)}\big).$$

4 Extension to compound Poisson (Chapter 5)

We obtain some Gaussian concentration inequalities of the approximation error of the invariant distribution associated with an EDS driven by a compound Poisson with sub-Gaussian jumps. We use a scheme biased on the one introduced by Panloup for pour more general Lévy processes.

With D. Loukianova and A. Gloter (LaMME, UEVE), we extended the techniques developed in the previous chapters, [GHL18]. This work has been submitted. We obtain some non-asymptotic concentration inequalities for dynamics with jumps

$$d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dW_t + \kappa(\mathbf{X}_t)d\mathbf{Z}_t, \tag{4.1}$$

where κ is a Lipschitz continuous function, $\mathbf{Z}_t$ is a square integrable Lévy process and π is the associated Lévy measure. An extension of the algorithm of Lamberton Pagès in this framework is introduced by Panloup [Pan08a]. The auteur establishes the convergence of the scheme and the associated CLT, cf. [Pan08b]:

$$\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi) \xrightarrow[n\to+\infty]{(\mathrm{loi})} \mathcal{N}\left(0, \int_{\mathbb{R}^d} \big(|\sigma^*\nabla\varphi|^2(x) + \int_{\mathbb{R}^d} |\varphi(x+\kappa(x)y) - \varphi(x)|^2\pi(dy)\big)\nu(dx)\right).$$

The jumps in the dynamic (4.1) yield a different limit variance than the one in the Brownian framework (2.4). We observe also that these jumps generate many problems, in particular the lack of integrability of the big jumps. Therefore, we focus on compound Poisson processes with sub-Gaussian jumps.

$$\mathbf{Z}_t = \sum_{k=1}^{N_t} Y_k, \tag{4.2}$$

where N_t is a compound Poisson of intensity 1 and $(Y_k)_{k\geqslant 1}$ is an i.i.d. sequence of sub-Gaussian random variables (satisfying **(GC)**). Thanks to the scheme with decreasing time step, we approximate the jump increments of the compound Poisson process $\Delta\mathbf{Z}_t$ by a random variable of the type B_nY_n where B_n follows a Bernoulli law such that $\mathbb{P}[B_n = 1] = \gamma_n$, $\mathbb{P}[B_n = 1] = 1 - \gamma_n$, and

$$X_{n+1} = X_n + \gamma_{n+1}b(X_n) + \sqrt{\gamma_{n+1}}\sigma(X_n)U_{n+1} + \kappa(X_n)Z_{n+1}. \tag{4.3}$$

This approximation allows us to apply the concentration method developed in the Brownian case: $\forall n \geqslant 1$, $0 < a \ll \frac{\sqrt{\Gamma_n}}{\sum_{k=1}^n \gamma_k^2}$,

$$\mathbb{P}\big[|\sqrt{\Gamma_n}\nu_n(\mathcal{A}\varphi)| \geqslant a\big] \leqslant 2\, C_n \exp\big(-c_n \frac{a^2}{2((1+r)\|\kappa\|_\infty^2\|\nabla\varphi\|_\infty^2 + \|\sigma\|_\infty^2\|\nabla\varphi\|_\infty^2)}\big),$$

with $c_n, C_n > 0$ converging towards 1 when $n \to +\infty$. The concentration constant is deteriorated by a multiplicative constant. Here, we do not obtain only an upper-bound depending on the uniform norms of the functions φ, σ and κ, but $(1+r)$ times the jump part. This is due to the difficulty to check that $(Z_n)_{n\geqslant 1}$ satisfies **(GC)**. We approximate a process $(\mathbf{Z}_t)_{t\geqslant 0}$ which does not have the property **(GC)**, by a sequence of random variables $(Z_n)_{n\geqslant 1}$ which has.

To non-asymptotically control the SDEs driven by less integrable Lévy processes than $\mathbf{Z}_t$ defined in (4.2), we need to use an other approach, for instance biased on a spectral theory for inhomogeneous schemes. Some Edgeworth or Berry-Esseen expansions for $\mathbf{Z}_t \in L^3$ might be performed.

II) Regularization by a degenerate noise
(Part III of the thesis)

Foreword

After a short introduction, in Section 5, I heuristically recall what is a regularization by a noise. Next, I briefly give some details on the Schauder estimates for non-degenerate equations. In Section 6, I present the method developed in my thesis.

For the sack of simplicity, I first show the techniques in the *kinetic* case (degenerate chain of ordre 2). Then, in Section 7, I present the model of degenerate chain. In this section, I state some existing results of Schauder estimates for the linear degenerate chain. I also give the main result of Chapter 6. In Section 8, I introduce some issues about uniqueness of solutions of SDEs. I state the main result of Chapter 7 of my thesis: the strong uniqueness for the degenerate chain of SDEs. I briefly explain Zvonkin Veretennikov transform which is the backbone of the analysis.

5 Regularization by a noise

This part is a collaboration with P.-E. Chaudru de Raynal (LAMA, Université de Savoie Mont Blanc) and S. Menozzi. First, we consider a degenerate Kolmogorov equation and we establish associated Schauder estimates for a minimal smoothness of the coefficients. It is an extension of the works of Lunardi [Lun97] and of Priola [Pri09] in a non-linear degenerate case. We introduce a new approach, even in a non-degenerate framework, for Schauder estimates. This is a constructive method biased on a perturbative technique. This work is submitted, see [CDRHM18a].

With this technique, we also establish the strong uniqueness for degenerate chain SDEs. The analysis relies on Zvonkin Veretennikov transform which requires some sophisticated knowledge of the associated Kolmogorov equation. This work is also submitted, [CDRHM18b].

5.1 Heuristic

First of all, I heuristically explain how regularization by a noise restores uniqueness of the solution of a scalar ODE of the kind:

$$\dot{x}_t = \mathbf{F}(t, x_t), \tag{5.1}$$

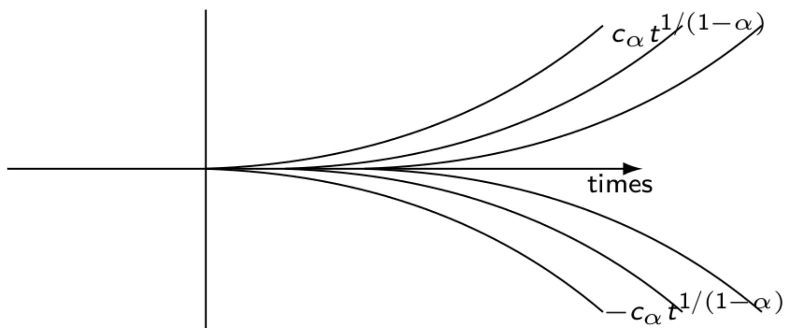
where $\mathbf{F}$ is continuous function. The Cauchy-Peano theorem ensures the existence of a solution x_t of (5.1). If $\mathbf{F}$ if Lipschitz continuous the Cauchy-Lipschitz theorem yields that the solution is unique. Nevertheless, if $\mathbf{F}$ is only Hölder continuous, we do not know *a priori* if the solution is unique. There are some counterexamples in this case. Let us consider the Peano's example,

$$\dot{x}_t = \text{sign}(x_t)|x_t|^\alpha, \alpha \in (0,1).$$

In this case, there is an infinite number of solutions:

$$x_t = c_\alpha (t - t_0)_+^{\frac{1}{1-\alpha}}, t_0 \in [0, T].$$

The picture below shows different maximal solutions of this ODE.



To have a unique solution of an ODE like (5.1), where $\mathbf{F}$ is Hölder continuous, we add a self-similar noise $(Z_t)_{t \geqslant 0}$, such as a Brownian motion, a stable process, a fractional Brownian motion...

$$dX_t = \mathbf{F}(t, X_t)dt + dZ_t. \tag{5.2}$$

We say that Z_t is γ self-similar if for any $a > 0$ with we have the following equality in law:

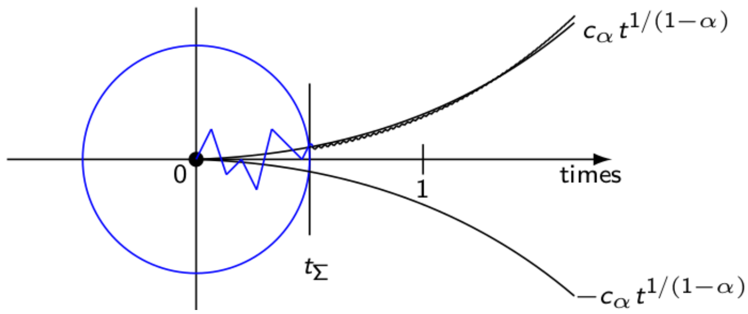
$$(Z_{at}, t \geqslant 0) \overset{\text{(loi)}}{=} (a^\gamma Z_t + (1 - a^\gamma) Z_0, t \geqslant 0). \tag{5.3}$$

In particular, if Z_t is Brownian motion, $\gamma = 1/2$, if Z_t is a α-stable process $\gamma = 1/\alpha$, or if Z_t is a fractional Brownian motion $\gamma = H$ where H is the associated Hurst exponent.

With the noise, ODE (5.1) becomes SDE (5.2) where there are two kinds of regime. Indeed Delarue and Flandoli [DF14] showed that there is a typical time $0 < t_\Sigma < 1$ such that:

- for any $t < t_\Sigma$ the variations of the noise dominates in the SDE. Somehow, the solution of SDE (5.2) leaves singularities of the drift $\mathbf{F}$.

- for any $t < t_\Sigma$ the drift dominates in (5.2). The solution of the SDE fluctuates around to the maximal solution of the associated ODE (5.1).



We then see heuristically that a path of the noise, until the moment t_Σ, forces the choice of the solution of the associated ODE. In other words, a self-similar noise restores the uniqueness (in some sense) of the solution of the ODE with noise.

For this phenomenon to happen the drift $\mathbf{F}$ cannot be too much irregular with respect to the self-similarity index of the noise. For instance, let us consider the kinetic Brownian diffusion *

$$\begin{aligned} dX_t^1 &= dW_t \rightsquigarrow t^{\frac{1}{2}} \\ dX_t^2 &= X_t^1 + \mathbf{F}_2(t, X_t^2)dt \rightsquigarrow t^{\frac{3}{2}}, \end{aligned}$$

with $\mathbf{F}_2$ α-Hölder in space. The non-degenerate variable X_t^1 is the Brownian motion, the standard deviation is $t^{\frac{1}{2}}$. While in the second level, the degenerate part, X_t^2 is homogeneous to the integral of X_t^1 then to the integral of the Brownian motion which is homogeneous to $t^{\frac{3}{2}}$.

If we consider the degenerate part, the self-similar noise of (5.2) corresponds to X_t^1 and satisfies to the property (5.3) for $\gamma = 3/2$. The fluctuations of the noise, before the critical time t_Σ, are more substantial than ones of the maximal solution of ODE (5.1) if the following condition holds

$$t_\Sigma^{\frac{1}{1-\alpha}} < t_\Sigma^{\frac{3}{2}} \Leftrightarrow 1 - \alpha < \frac{2}{3} \Leftrightarrow \alpha > \frac{1}{3}. \tag{5.4}$$

We then see, in the kinetic case, the threshold for the weak uniqueness established by Chaudru de Raynal and Menozzi [CdRM17]. This required regularity is minimal because, in this article, the authors show a counter-example of non-uniqueness for a drift α-Hölder continuous with $\alpha < 1/3$.

These thresholds are the same as those found for Schauder estimates associated with solution Kolmogorov equation, cf. Section 7 further. Nevertheless, these thresholds of minimal regularity are different for strong uniqueness, see Section 8 below.

5.2 Non-degenerate Schauder estimates

Schauder estimates are crucial in (numerous) fields which require some precise controls of the solution of PDEs. For example, it is important in the study the discretization error of SDEs, as seen in the first part of this summary.

Here, I briefly recall some existing results concerning Schauder estimates for some non-degenerate Kolmogorov equations. For any finite horizon $T > 0$, we consider the following Cauchy problem:

*We say kinetic or speed/position, because after integrating the equation, we see that X_t^2 corresponds to the integral of X_t^1. Hence X_t^1 (the speed) can be regarded as the derivative of X_t^2 (the position).

$\forall (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$,

$$\begin{cases} \partial_t u(t,x) + \langle \mathbf{F}(t,x), \mathbf{D}u(t,x) \rangle + \frac{1}{2}\mathrm{Tr}\big(\mathbf{D}^2 u(t,x) a(t,x)\big) = -f(t,x), \\ u(T,x) = g(x),\ x \in \mathbb{R}^d, \end{cases} \tag{5.5}$$

a is U.E. and bounded.

$\star$ Friedman [Fri64]: for $\mathbf{F}, \sigma \in C_b^{\frac{\gamma}{2},\gamma}$, i.e. $\sigma, \mathbf{F}$ bounded, $\frac{\gamma}{2}$-Hölder continuous in time and γ-Hölder in space, the solution u of equation (5.5) satisfies:

$$\|u\|_{C_b^{\frac{2+\gamma}{2},2+\gamma}} \leqslant C(\|f\|_{C_b^{\frac{\gamma}{2},\gamma}} + \|g\|_{C_b^{2+\gamma}}), \tag{5.6}$$

where the norms are the standard Hölder norms in time/space.

$\star$ When the drift is unbounded in space, Schauder estimates was for a long time an open problem. It was solved in 2010 by Krylov and Priola [KP10]: for $\sigma \in C_b^{\frac{\gamma}{2},\gamma}$, and $\mathbf{F} \in L^\infty(C^\gamma))$, i.e. $\mathbf{F}$ bounded in time and γ-Hölder continuous in space, the solution u of a PDE of the type (5.5) with a potential term satisfies:

$$\|u\|_{L^\infty(C_b^{2+\gamma})} \leqslant C(\|f\|_{L^\infty(C^\gamma)} + \|g\|_{C_b^{2+\gamma}}), \tag{5.7}$$

the norms $\|\cdot\|_{L^\infty(C_b^{2+\gamma})}$, $\|\cdot\|_{L^\infty(C^\gamma)}$ are the uniform norms in time and Hölder in space.

In particular, these estimates ensure the uniqueness of the solution of the considered PDE. Indeed, if there are two solutions u_1 and u_2 satisfying the same Schauder then:

$$\|u_1 - u_2\|_{L^\infty(C_b^{2+\gamma})} \leqslant 0 \implies u_1 = u_2.$$

We also see the so-called "parabolic bootstrap":

- in space, if the source function $f \in C_b^\gamma$ then the solution $u \in C_b^{2+\gamma}$,
- in time, if the source function $f \in C_b^{\frac{\gamma}{2}}$ then the solution $u \in C_b^{\frac{2+\gamma}{2}}$.

The regularity gain of the solution compared with the source function f is done according to the associated parabolic scale. Indeed, in the non-degenerate case, the associated parabolic pseudo-distance is

$$\mathbf{d}_P\big((t,x),(t',x')\big) = |t-t'|^{1/2} + |x-x'|. \tag{5.8}$$

From an other point of view, we can see this parabolic scale through the typical differential operator $\mathscr{L}_0$ associated with (5.5) and defined by:

$$\mathscr{L}_0 := \partial_t + \frac{1}{2}\Delta.$$

For any $\lambda > 0$ the dilatation operator $\delta_\lambda : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \mapsto \delta_\lambda(t,x) = \big(\lambda^2 t, \lambda x\big)$, we have:

$$\mathscr{L}_0 v = 0 \implies \mathscr{L}_0(v \circ \delta_\lambda) = 0. \tag{5.9}$$

In other words, the introduced pseudo-distance in (5.8), makes the dilatation operator δ_λ homogeneous in time/space. This time/space scale is intrinsic to the considered parabolic problem.

The regularity gain shown by Schauder estimates (5.6), (5.7) can be regarded as an expression of regularity by noise. Indeed, Kolmogorov equation (5.5) is linked to a SDE of the type (5.2). In particular, the l.h.s. term in (5.5) corresponds to the non-martingale part in the Itô's formula. The term of second order operator in equation (5.5) is the deterministic counterpart of the presence of a self-similar noise with 1/2 as self-similarity index, here the Brownian motion.

6 Perturbative method

One of the main contributions in the second part of my thesis was a new method developed to establish Schauder estimates. This method is constructive unlike the continuity method, and is robust. Recently, our method was used in [CdRMP19] to establish some Schauder estimates for Kolmogorov equation associated with fractional operator (for a stable process) in a supercritical regularity and self-similarity framework, namely when the self-similarity index of the noise does not dominate the order 1 of the drift, $\alpha < 1$.

6.1 Kinetic model

For the sack of simplicity, in this section, I explain our method in a kinetic framework. This model is used in different fields, in finance with dynamics of Asian options, in Hamiltonian physic... The full degenerate chain will be dealt in the following section. Consider the Cauchy problem associated with the following kinetic model:

$$\begin{aligned} d\mathbf{X}_t^1 &= \mathbf{F}_1(t,\mathbf{X}_t) + \sigma(t,\mathbf{X}_t)dW_t,\\ d\mathbf{X}_t^2 &= \mathbf{F}_2(t,\mathbf{X}_t)dt, \end{aligned} \tag{6.1}$$

σ is a square root of a and where $D_{\mathbf{x}_1}\mathbf{F}_2(t,\mathbf{x})$ is non-degenerate for all t and $\mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2) \in \mathbb{R}^{2d}$. The non-degeneracy condition of the sub-diagonal of the Jacobian matrix of the drift corresponds to the weak Hömander's assumption (successive Lie brackets associated to a regularised version of (6.1) span $\mathbb{R}^{2d}$, cf. [Hör67]) also called Kalman's rank condition (cf. [Zab08], Chapitre 1). This hypothesis then ensures the existence of a solution of SDE (6.1) and of the associated PDE which is: $\forall T > 0,\ (t,\mathbf{x}) \in [0,T] \times \mathbb{R}^{nd}$,

$$\begin{cases} \partial_t u(t,\mathbf{x}) + \langle \mathbf{F}(t,\mathbf{x}), \mathbf{D}u(t,\mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big(D^2_{\mathbf{x}_1}u(t,\mathbf{x})a(t,\mathbf{x})\big) = -f(t,\mathbf{x}),\\ u(T,\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2) \in \mathbb{R}^{2d}, \end{cases} \tag{6.2}$$

where the drift is such that $\mathbf{F}(t,\mathbf{x}) := (\mathbf{F}_1(t,\mathbf{x}),\mathbf{F}_2(t,\mathbf{x}))$. The associated differential operator is defined by

$$Lu(t,\mathbf{x}) := \langle \mathbf{F}(t,\mathbf{x}), \mathbf{D}u(t,\mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big(D^2_{\mathbf{x}_1}u(t,\mathbf{x})a(t,\mathbf{x})\big). \tag{6.3}$$

We suppose also that:

- $(\mathbf{S}_a)$ $a \in L^\infty(C_b^{\gamma,\frac{\gamma}{3}})$, bounded, γ-Hölder continuous in $\mathbf{x}_1$ and $\frac{\gamma}{3}$-Hölder in $\mathbf{x}_2$,
- $(\mathbf{S}_{\mathbf{F}_1})$ $\mathbf{F}_1 \in L^\infty(C^{\gamma,\frac{\gamma}{3}})$, bounded in time, γ-Hölder in $\mathbf{x}_1$ and $\frac{\gamma}{3}$-Hölder continuous in $\mathbf{x}_2$,
- $(\mathbf{S}_{\mathbf{F}_2})$ $\mathbf{F}_2 \in L^\infty(C^{1+\gamma,\frac{1+\gamma}{3}})$, bounded in time, $1+\gamma$-Hölder continuous in $\mathbf{x}_1$ and $\frac{1+\gamma}{3}$-Hölder continuous in $\mathbf{x}_2$,

- $(\mathbf{S}_f)$ $f \in L^\infty(C_b^{\gamma,\frac{\gamma}{3}})$, bounded, γ-Hölder continuous in $\mathbf{x}_1$ and $\frac{\gamma}{3}$-Hölder continuous in $\mathbf{x}_2$,
- $(\mathbf{S}_g)$ $g \in C_b^{2+\gamma,\frac{2+\gamma}{3}})$, bounded, $2+\gamma$-Hölder continuous in $\mathbf{x}_1$ and $\frac{2+\gamma}{3}$-Hölder continuous in $\mathbf{x}_2$,

We want to show that the solution u of the Cauchy problem (6.2) satisfies:

$$\|u\|_{L^\infty(C_b^{2+\gamma,\frac{2+\gamma}{3}})} \leqslant C(\|f\|_{L^\infty(C^{\gamma,\frac{\gamma}{3}})} + \|g\|_{C_b^{2+\gamma,\frac{2+\gamma}{3}}}). \tag{6.4}$$

The expected regularity gain is relied on the intrinsic scale of the problem (6.2). Indeed, for the typical differential operator associated with (6.2) $\mathscr{L}_0' := \partial_t + \frac{1}{2}\Delta_{\mathbf{x}_1}$, and the dilatation operator

$$\delta_\lambda' : (t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^{2d} \mapsto \delta_\lambda(t,\mathbf{x}) = \big(\lambda^2 t, \lambda \mathbf{x}_1, \lambda^3 \mathbf{x}_2\big), \tag{6.5}$$

we also obtain similarly to (5.9):

$$\mathscr{L}_0' v = 0 \implies \mathscr{L}_0'(v \circ \delta_\lambda') = 0.$$

6.2 The Choice of *proxy*

To ensure existence and uniqueness of the various function used in the following, the first step is mollifying all the coefficients of equation (6.2). The goal is to establish that the solution u_m of the mollified version of (6.2) satisfies Schauder estimates (6.4) uniformly in m, the mollification coefficient. For the sack of notation simplicity, we will omit the index m.

Next, we linearise the equation around a well-known *proxy* and control the approximation error. For any operator $\tilde{L}$ associated with a *proxy* $\tilde{\mathbf{X}}_t$ which has a probabilistic density $\tilde{p}(t,s,\mathbf{x},\mathbf{y})$ we write

$$\begin{cases} \partial_t u(t,\mathbf{x}) + \tilde{L}u(t,\mathbf{x}) = -f(t,\mathbf{x}) - (L-\tilde{L})u(t,\mathbf{x}),\ (t,\mathbf{x}) \in [0,T) \times \mathbb{R}^{nd}, \\ u(T,\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^{nd}. \end{cases} \tag{6.6}$$

Thanks to the Duhamel's formula, corresponding to a *parametrix* expansion at the first order, one has

$$u(t,\mathbf{x}) = \tilde{P}_{T,t} g(\mathbf{x}) + \tilde{G}f(t,\mathbf{x}) + \int_t^T ds \int_{\mathbb{R}^{nd}} \tilde{p}(t,s,\mathbf{x},\mathbf{y})(L-\tilde{L})u(s,\mathbf{y})d\mathbf{y}, \tag{6.7}$$

where we respectively define the *semi-group* and the *Green kernel* by:

$$\tilde{P}_{T,t} g(\mathbf{x}) := \int_{\mathbb{R}^{nd}} \tilde{p}(t,T,\mathbf{x},\mathbf{y}) g(\mathbf{y}) d\mathbf{y}, \quad \tilde{G}f(t,\mathbf{x}) := \int_t^T ds \tilde{P}_{s,t} f(s,\mathbf{x}). \tag{6.8}$$

The last contribution in equation (6.7), $\int_t^T ds \int_{\mathbb{R}^{nd}} \tilde{p}(t,s,\mathbf{x},\mathbf{y})(L-\tilde{L})u(s,\mathbf{y})d\mathbf{y}$ is a remainder term.

The *proxy* choice is crucial. On the one hand, the maximum smoothness of the solution that we can expect is the one of the semi-group and the Green Kernel associated with the *proxy.* On the other hand, we need to find a *proxy* close enough from the considered process in order to make the remainder term negligible.

A natural choice of the *proxy* is a linearisation of the process (6.1) around the flow associated with the drift. In other words, let us consider the flow

$$\dot{\boldsymbol{\theta}}_{v,\tau}(\boldsymbol{\xi}) = \mathbf{F}(v, \boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi})),\ v \in [\tau, T],\ \boldsymbol{\theta}_{\tau,\tau}(\boldsymbol{\xi}) = \boldsymbol{\xi}^*, \tag{6.9}$$

*The solution of this ODE is unique for the mollified version of $\mathbf{F}$ by Cauchy-Lipschitz theorem.

where $(\tau, \boldsymbol{\xi}) \in [0,T] \times \mathbb{R}^{nd}$ are freezing parameters.

The choice of this parameters depends on the analysis strategy. Indeed, if we choose $\tau = s$, $\boldsymbol{\xi} = \mathbf{y}$ with the notations from (6.8), then it is a *backward proxy*. The *proxy* starts from $\mathbf{y}$ at moment s and goes to $\mathbf{x}$ at the moment $t < s$. This approach, introduced by McKean and Singer [MS67], is useful for density estimates, see also Delarue and Menozzi [DM10], and Menozzi *et al.* [KMM10], [CdRM17][†].

This method allows us to use some centering arguments but is not convenient to use *cancellation* techniques [‡] because the density associated with the *proxy* depends on $\mathbf{y}$ which is the current variable of integration in the Duhamel's formula (6.7), i.e. $\tilde{p}(t,s,\mathbf{x},\mathbf{y}) = \tilde{p}^{(s,\mathbf{y})}(t,s,\mathbf{x},\mathbf{y})$. in other words, the *proxy* density is not a probabilistic density. One does not a priori have $\int_{\mathbb{R}^{2d}} \tilde{p}^{(s,\mathbf{y})}(t,s,\mathbf{x},\mathbf{y})d\mathbf{y} \neq 1$. Controlling higher-order derivatives than the gradient is hard.

To bypass this difficulty, we used a *forward* approach. The *proxy* starts at $\mathbf{x}$ in t and goes to $\mathbf{y}$ at the moment s. In other words, we take $\tau = t$, $\boldsymbol{\xi} = \mathbf{x}$. This method introduced by Il'in, Kalashnikov and Oleinik [IKO62], was also used by Chaudru de Raynal [CdR17] for the kinetic case (6.1).

6.3 Properties of the *proxy*

With this choice of flow $\big(\boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi})\big)_{v\in[\tau,T]}$, we linearise the SDE around it by a Taylor expansion at the first order. For any $v \in [t,s]$

$$\begin{aligned} d\tilde{\mathbf{X}}_v^{(\tau,\boldsymbol{\xi})} &= [\mathbf{F}(v,\boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi})) + D\mathbf{F}(v,\boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi}))(\tilde{\mathbf{X}}_v^{(\tau,\boldsymbol{\xi})} - \boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi}))]dv + B\sigma(v,\boldsymbol{\theta}_{v,\tau}(\boldsymbol{\xi}))dW_v, \\ \tilde{\mathbf{X}}_t^{(\tau,\boldsymbol{\xi})} &= \mathbf{x}, \end{aligned} \tag{6.10}$$

with $B = \begin{pmatrix} \mathbf{I}_{d,d} & \mathbf{0}_{d,d} \\ \mathbf{0}_{d,d} & \mathbf{0}_{d,d} \end{pmatrix}$, and for any $\mathbf{z} \in \mathbb{R}^{nd}$: $D\mathbf{F}(v,\mathbf{z}) = \begin{pmatrix} \mathbf{0}_{d,d} & \mathbf{0}_{d,d} \\ D_{\mathbf{z}_1}\mathbf{F}_2(v,\mathbf{z}) & \mathbf{0}_{d,d} \end{pmatrix}$ (corresponding to the variations of the drift according to the variable which transmits the noise). We suppose that $D_{\mathbf{z}_1}\mathbf{F}_2(v,\mathbf{z})$ is invertible, i.e. the weak Hörmander condition is satisfied.

SDE (6.10) being linear, the process $\tilde{\mathbf{X}}_s^{(\tau,\boldsymbol{\xi})}$ has a Gaussian density. The covariance denoted by $\tilde{\mathbf{K}}_{v,t}^{(\tau,\boldsymbol{\xi})}$ has a "good scaling" property,

$$\tilde{\mathbf{K}}_{v,t}^{(\tau,\boldsymbol{\xi})} \asymp (v-t)^{-1}\mathbb{T}_{v-t}^2 \,{}^{\S}. \tag{6.11}$$

where for any $s > 0$, we define the intrinsic scale matrix by: $\mathbb{T}_s = \begin{pmatrix} s\mathbf{I}_{d,d} & \mathbf{0}_{d,d} \\ \mathbf{0}_{d,d} & s^2\mathbf{I}_{d,d} \end{pmatrix}$. This scale matrix and identity (6.11) rely on the behaviour of the dilatation operator defined in (6.5).

[†] and [Men11], [Men18].

[‡] The *cancellation* method consists in writing for a function h such that $\int h(y)dy = 0$, and for a smooth function f, $\int h(x-y)f(y)dy = \int h(x-y)[f(y)-f(x)]dy$, and we take advantage of the smoothness of f.

[§] i.e. $\exists C \geqslant 1$ such that for all $0 \leqslant t < v \leqslant T < 1$:

$$\forall \boldsymbol{\zeta} \in \mathbb{R}^{nd},\ C^{-1}(v-t)^{-1}|\mathbb{T}_{v-t}\boldsymbol{\zeta}|^2 \leqslant \langle \tilde{\mathbf{K}}_{v,t}^{(\tau,\boldsymbol{\xi})}\boldsymbol{\zeta},\boldsymbol{\zeta}\rangle \leqslant C(v-t)^{-1}|\mathbb{T}_{v-t}\boldsymbol{\zeta}|^2,$$

The *proxy* density $\tilde{p}^{(\tau,\boldsymbol{\xi})}(t,s,\mathbf{x},\mathbf{y})$ has the following property: there is $C>0$ such that for each $i \in \{1,2\}$

$$|D_{\mathbf{x}_i}\tilde{p}^{(\tau,\boldsymbol{\xi})}(t,s,\mathbf{x},\mathbf{y})| \leqslant \frac{C}{(s-t)^{(i-1/2)}}\bar{p}^{(\tau,\boldsymbol{\xi})}_{C^{-1}}(t,s,\mathbf{x},\mathbf{y}), \tag{6.12}$$

where $\bar{p}^{(\tau,\boldsymbol{\xi})}_{C^{-1}}(t,s,\mathbf{x},\mathbf{y})$ is a Gaussian density $\mathcal{N}\big(\mathbf{m}^{\tau,\boldsymbol{\xi}}_{s,t}(\mathbf{x}),(s-t)^{-1}\mathbb{T}^2_{s-t}\big)$, $\mathbf{m}^{\tau,\boldsymbol{\xi}}_{s,t}(\mathbf{x}) \in \mathbb{R}^d$¶. This Gaussian density $\bar{p}^{(\tau,\boldsymbol{\xi})}_{C^{-1}}(t,s,\mathbf{x},\mathbf{y})$ is actually the heat kernel associated with the pseudo-distance

$$\mathbf{d}\big(\mathbf{x},\mathbf{x}'\big) = |\mathbf{x}_1-\mathbf{x}'_1| + |\mathbf{x}_2-\mathbf{x}'_2|^{1/3}, \tag{6.13}$$

which corresponds to the homogeneous dilatation operator (6.5).

We can see that, in view of (6.12), differentiating in space the density of the *proxy* yields time singularities. The main issue of the analysis consists in making these singularities integrable.

The tilde «˜» concerns the *proxy* and the bar «¯» concerns the heat kernel associated with the pseudo-distance $\mathbf{d}$.

From now on, for the sack of simplicity, we omit the indexes C^{-1}, τ, $\boldsymbol{\xi}$. We choose the freezing parameters after potential differentiations of the Duhamel's formula (6.7). Let us note that taking $(\boldsymbol{\xi},\tau) = (\mathbf{x},t)$ yields that the mean becomes equal to the flow, i.e. $\mathbf{m}^{(\tau,\boldsymbol{\xi})}_{v,t}(\mathbf{x}) = \boldsymbol{\theta}_{v,t}(\mathbf{x})$. In the following, we take $\tau = t$. We also take $\boldsymbol{\xi} = \mathbf{x}$ to control the uniform norms. For the Hölder norms control, the choice of *proxy* will be more subtle.

6.4 Controls of uniform norms

The controls of uniform norms of the semi-group, the Green kernel and their derivatives in $\mathbf{x}_1$ are made by *cancellation* techniques. To illustrate this method, let us consider the second derivative of the term of order 2 in the remainder term in the Duhamel's formula (6.7).

$$\begin{aligned}
&\int_t^T ds \int_{\mathbb{R}^{2d}} d\mathbf{y} D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})\frac{1}{2}\mathrm{Tr}\big([a(s,\mathbf{y})-a(s,\boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi}))]D^2_{\mathbf{y}_1}u(s,\mathbf{y})\big)\\
\overset{(\mathbf{S}_a)+(6.12)}{\leqslant}\ & \frac{C}{2}\|a\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\|\|D^2_{\mathbf{y}_1}u\|_{L^\infty}\int_t^T ds\int_{\mathbb{R}^{2d}} d\mathbf{y}\frac{\bar{p}(t,s,\mathbf{x},\mathbf{y})}{s-t}\big(|\mathbf{y}_1-\boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi})_1|^\gamma + |\mathbf{y}_2-\boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi})_2|^{\frac{\gamma}{3}}\big)\\
\leqslant\ & C\|a\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\|\|D^2_{\mathbf{y}_1}u\|_{L^\infty}\int_t^T ds\int_{\mathbb{R}^{2d}} d\mathbf{y}\frac{\exp\Big(-C^{-1}(s-t)\big|\mathbb{T}^{-1}_{s-t}\big(\mathbf{y}-\boldsymbol{\theta}_{s,t}(\mathbf{x})\big)\big|^2\Big)}{(s-t)^{1-\frac{\gamma}{2}+\frac{n^2d}{2}}}\\
\leqslant\ & C(T-t)^{\gamma/2}\|a\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\|D^2_{\mathbf{y}_1}u\|_{L^\infty}.
\end{aligned}\tag{6.14}$$

The penultimate inequality, comes from the following property: for any $\ell>0$, there is $C_\ell \in (0,1)$ such that for any $x>0$

$$x^\ell e^{-x^2} \leqslant C_\ell^{-1}e^{-C_\ell x^2}. \tag{6.15}$$

In order to control the uniform norms by a circular argument * we suppose that the horizon T is small enough.

¶The mean $\mathbf{m}^{\tau,\boldsymbol{\xi}}_{s,t}(\mathbf{x})$ is explicitly defined according to the resolvent associated with the drift $\mathbf{F}$. For more informations see Section 3.1 of Chapter 6 or Section 2.1 of Chapter 7 of my thesis.

*Let us recall that a circular argument is: "if $x \leqslant cx+y$ with $c<1$ and $y>0$ then $x \leqslant \frac{1}{1-c}y$".

Nevertheless, we did not control the degenerate part of the remainder term in the Duhamel's formula (6.7) yet, which is more delicate. Indeed, this contribution involves the derivatives of the solution u according to the degenerate directions ($D_{\mathbf{y}_2}u(s,\mathbf{y})$). But the non-mollified version of u is not *a priori* differentiable according to the degenerate directions. Because, in view of (6.4), we expect u to be $(2+\gamma)/3 < 1$ Hölder continuous in the variable $\mathbf{x}_2$. To bypass this problem, we write

$$\Delta_{\mathbf{F}}(\tau, s, \boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}), \mathbf{y}) := \Big(\mathbf{F}_2(s,\mathbf{y}) - \mathbf{F}_2(s, \boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi})) - D_{\mathbf{x}_1}\mathbf{F}_2(s, \boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi}))(\mathbf{y} - \boldsymbol{\theta}_{s,\tau}(\boldsymbol{\xi}))_1\Big), \tag{6.16}$$

and by an integration by parts

$$\begin{aligned}&\int_t^T ds \int_{\mathbb{R}^{2d}} d\mathbf{y} D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})\langle \Delta_{\mathbf{F}}(\tau, s, \boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}), \mathbf{y}), D_{\mathbf{y}_i}u(s,\mathbf{y})\rangle \\ &= -\int_t^T ds \int_{\mathbb{R}^{2d}} d\mathbf{y} D_{\mathbf{y}_2}\cdot \big[D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})\Delta_{\mathbf{F}}(\tau, s, \boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}), \mathbf{y})\big]u(s,\mathbf{y}).\end{aligned} \tag{6.17}$$

The key argument to control this term is to use some Besov duality properties. We identify the usual Hölder space to a Besov space: $C_b^{\tilde{\alpha}}(\mathbb{R}^d,\mathbb{R}) = B^{\tilde{\alpha}}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$, $\tilde{\alpha} \in (0,1)$.

A function f lies in the Besov space $B^s_{p,q}(\mathbf{R})$, $1 \leqslant p,q \leqslant +\infty$, $s \in \mathbb{R}$ if

$$f \in W^{\lfloor s\rfloor, p}(\mathbf{R}), \text{ and } \int_0^\infty \left|\frac{\sup_{|h|\leqslant t}\|f(x-2h) - 2f(x-h) - f(x)\|_{L^p}}{t^\alpha}\right|^q \frac{dt}{t} < \infty,$$

for more characterisations of Besov spaces see [Tri83] †.

Furthermore, it is well known that the spaces $B^{\tilde{\alpha}_i}_{\infty,\infty}(\mathbb{R}^d,\mathbb{R})$ and $B^{-\tilde{\alpha}}_{1,1}(\mathbb{R}^d,\mathbb{R})$ are dual with $\tilde{\alpha} = \frac{2+\gamma}{3}$, cf. [LR02].

We show by *cancellation* techniques that

$$\big\|\mathbf{y}_2 \mapsto D_{\mathbf{y}_2}\cdot\big(D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})\Delta_{\mathbf{F}}(\tau,s,\boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}),\cdot)\big)\big\|_{B_{1,1}^{-\frac{2+\gamma}{3}(\mathbb{R}^d)}} \leqslant C(s-t)^{\frac{\gamma}{2}-1}\int_{\mathbb{R}^d}\bar{p}(t,s,\mathbf{x},\mathbf{y})d\mathbf{y}_2.$$

In the r.h.s. the time singularity is integrable. We integrate according to s and $\mathbf{y}_1$ in (6.17) and we obtain the same result as for the non-degenerate part.

$$\Big|\int_t^T ds\int_{\mathbb{R}^{2d}} d\mathbf{y} D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})\langle\Delta_{\mathbf{F}}(\tau,s,\boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}),\mathbf{y}), D_{\mathbf{y}_i}u(s,\mathbf{y})\rangle\Big| \leqslant C(T-t)^{\gamma/2}\|u\|_{L^\infty(C_b^{2+\gamma,\frac{2+\gamma}{3}})} \tag{6.18}$$

6.5 Control of Hölder modulus

To obtain some controls of Hölder norms, we have to handle with new difficulties. This can be already observed in the previous inequalities where there is a kind of "margin" in time of order γ. In inequalities (6.14) and (6.18), this margin is the multiplicative term $(T-t)^{\gamma/2}$, which is homogeneous to the intrinsic distance $\mathbf{d}^\gamma$. That corresponds to the maximal regularity of the solution, namely the γ-Hölder regularity of $D^2_{\mathbf{x}_1}u$. Hence, by homogeneity, to control the corresponding Hölder norm we cannot take advantage of this margin (in particular to use a circular argument).

†The chosen representation of Besov norms in my work is the thermic one.

To exploit the maximal regularity of the solution u, we have to be more subtle in the choice of freezing parameters. Estimating the γ-Hölder norm means that we show that for all points $\mathbf{x},\mathbf{x}'\in\mathbb{R}^{2d}$:

$$\sup_{t\in[0,T]}|D^2_{\mathbf{x}_1}u(t,\mathbf{x})-D^2_{\mathbf{x}_1}u(t,\mathbf{x}')|\leqslant C\mathbf{d}^\gamma(\mathbf{x},\mathbf{x}').$$

We write the Duhamel's formula (6.7) for these two points, and we obtain 4 associated freezing parameters, $(\tau,\boldsymbol{\xi})$ and $(\tau',\boldsymbol{\xi}')$, to calibrate.

For the choice of these parameters, we need to consider two possibilities: $\mathbf{x}$ and $\mathbf{x}'$ close from each other or not. This closeness is seen according to the difference between the running time and initial time t, for a constant $c_0>0$ (specified latter):

- the *off-diagonal* regime: $c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')>(s-t)$,
- and the *diagonal* regime: $c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')\leqslant(s-t)$,

• In the *off-diagonal* case, the points being far from each other, we cannot expect any regularity gain by choosing the same *proxy* point for $\mathbf{x}$ and $\mathbf{x}'$. The idea is to take $(\boldsymbol{\xi},\boldsymbol{\xi}')=(\mathbf{x},\mathbf{x}')$. To illustrate this technique, here we show how to control the Hölder norm of the second derivative of the Green kernel $D^2_{\mathbf{x}_1}\tilde{G}^{(\tau,\boldsymbol{\xi})}f$, defined in (6.8). By triangular inequality, we readily get:

$$\begin{aligned}&\Big|\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}[D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})-D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x}',\mathbf{y})]f(s,\mathbf{y})\Big|\\ &\leqslant\Big|\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})f(s,\mathbf{y})\Big|+\Big|\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x}',\mathbf{y})f(s,\mathbf{y})\Big|.\end{aligned}$$

Next, we control the first term in the r.h.s., the other one is handled similarly.

$$\begin{aligned}&\Big|\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})f(s,\mathbf{y})\Big|\\ \overset{(cancellation)}{=}\;&\Big|\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}D^2_{\mathbf{x}_1}\tilde{p}(t,s,\mathbf{x},\mathbf{y})[f(s,\mathbf{y})-f(s,\boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}))]\Big|\\ \overset{(\mathbf{S}_f)\ +(6.12)}{\leqslant}\;&\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}\frac{C}{s-t}\bar{p}(t,s,\mathbf{x},\mathbf{y})\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\mathbf{d}^\gamma(\mathbf{y},\boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}))\\ =\;&\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds\int_{\mathbb{R}^{2d}}d\mathbf{y}\frac{C}{(s-t)^{1-\frac{\gamma}{2}}}\bar{p}(t,s,\mathbf{x},\mathbf{y})\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\Big(\frac{\mathbf{d}(\mathbf{y},\boldsymbol{\theta}_{s,t}(\boldsymbol{\xi}))}{(s-t)^{1/2}}\Big)^\gamma\\ \overset{(6.15)}{\leqslant}\;&C\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\int_t^{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}ds(s-t)^{\frac{\gamma}{2}-1}=Cc_0^{\frac{\gamma}{2}}\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\mathbf{d}^\gamma(\mathbf{x},\mathbf{x}').\end{aligned}$$

We then obtain the desired result.

• Now, in the *diagonal* case, we take advantage of the closeness between $\mathbf{x}$ and $\mathbf{x}'$. To do that, we choose the same freezing points $(\boldsymbol{\xi},\boldsymbol{\xi}')=(\mathbf{x},\mathbf{x})$. We write next by a Taylor expansion at the first order for the *proxy* density:

$$
\begin{aligned}
& \Big|\int_{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}^{t} ds \int_{\mathbb{R}^{2d}} d\mathbf{y}[D^2_{\mathbf{x}_1}\tilde p(t,s,\mathbf{x},\mathbf{y}) - D^2_{\mathbf{x}_1}p^{(\tau,\boldsymbol{\xi}')}(t,s,\mathbf{x}',\mathbf{y})]f(s,\mathbf{y})\Big| \\
\leqslant \quad & \Big|\int_{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}^{t} ds \int_{\mathbb{R}^{2d}} d\mathbf{y} \int_0^1 d\lambda(\mathbf{x}-\mathbf{x}')\cdot \mathbf{D}_{\mathbf{x}} D^2_{\mathbf{x}_1}\tilde p(t,s,\mathbf{x}'+\lambda(\mathbf{x}-\mathbf{x}'),\mathbf{y})f(s,\mathbf{y})\Big| \\
\overset{(6.12)+(\text{cancellation})}{\leqslant} & C\|f\|_{L^\infty}\sum_{i=1}^2 \int_{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}^{t} ds \int_{\mathbb{R}^{2d}} d\mathbf{y} \underbrace{\big|(\mathbf{x}-\mathbf{x}')_i\big|}_{\leqslant \mathbf{d}^{2i-1}(\mathbf{x},\mathbf{x}')} \frac{C}{(s-t)^{1+i-\frac12-\frac{\gamma}{2}}}\bar p(t,s,\mathbf{x},\mathbf{y}) \\
\leqslant \quad & C\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\mathbf{d}^{2i-1}(\mathbf{x},\mathbf{x}')\int_{t+c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')}^{t} \frac{C ds}{(s-t)^{1+i-\frac12-\frac{\gamma}{2}}} \leqslant C c_0^{\frac{\gamma+1}{2}-i}\|f\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}\mathbf{d}^{\gamma}(\mathbf{x},\mathbf{x}').
\end{aligned}
$$

6.6 Conclusion

Our choice of freezing points $(\boldsymbol{\xi},\boldsymbol{\xi}')$, according to the regime *diagonal* or *hors-diagonal*, depends on the variable of integration s. This "*cut locus*" technique yields a new contribution corresponding to a discontinuous term, to control in the Duhamel's formula (6.7)

$$
\tilde P^{(\tau,\boldsymbol{\xi}')}_{t_0,t}u(t_0,\mathbf{x}') - \tilde P^{(\tau,\tilde{\boldsymbol{\xi}}')}_{t_0,t}u(t_0,\mathbf{x}')\Big|_{(\boldsymbol{\xi}',\tilde{\boldsymbol{\xi}}')=(\mathbf{x}',\mathbf{x})}, \tag{6.19}
$$

with $t_0 := t + c_0\mathbf{d}^2(\mathbf{x},\mathbf{x}')$ (the semi-group $\tilde P^{(\cdot,\cdot)}_{\cdot,\cdot}$ is defined in (6.8)). A similar analysis that the one for the control of $\tilde P_{T,t}g(\mathbf{x})$ allows us to estimate the discontinuous term.

We then obtain Schauder estimates (6.4) if the Hölder modulus of the coefficients are small enough, $\exists 0<\Lambda = \Lambda\big(\|a\|_{L^\infty(C_b^{\gamma,\frac{\gamma}{3}})}, \|\mathbf{F}_1\|_{L^\infty(C^{\gamma,\frac{\gamma}{3}})}, \|\mathbf{F}_2\|_{L^\infty(C^{1+\gamma,\frac{1+\gamma}{3}})}\big)$ such that

$$
\|u\|_{L^\infty(C_b^{2+\gamma,\frac{2+\gamma}{3}})} \leqslant C\Big(\|f\|_{L^\infty(C^{\gamma,\frac{\gamma}{3}})} + \|g\|_{C_b^{2+\gamma,\frac{2+\gamma}{3}}} + \|u\|_{L^\infty(C_b^{2+\gamma,\frac{2+\gamma}{3}})}\Lambda\big[1+c_0^{\frac{\gamma}{2}}+T^{\frac{\gamma}{2}}\big]\Big).
$$

If $\Lambda\big[1+c_0^{\frac{\gamma}{2}}+T^{\frac{\gamma}{2}}\big]<1$, we conclude thanks to a circular argument. To avoid the condition the Hölder modulus of a and $\mathbf{F}$ are small, we use "scaling" method with a similar dilatation as in (6.5).

For any final time T, we repeat the previous method a sufficient number of times with as the terminal condition the solution of the PDE.

$$
\begin{cases}
\partial_t u_k(t,\mathbf{x}) + \langle \mathbf{F}(t,\mathbf{x}), \mathbf{D}u_k(t,\mathbf{x})\rangle + \frac12 \mathrm{Tr}\big(D^2_{\mathbf{x}_1}u_k(t,\mathbf{x})a(t,\mathbf{x})\big) = -f(t,\mathbf{x}),\ t,\in [T\frac{N-k}{N}, T\frac{N+1-k}{N}),\\
u_k((1-\frac{k-1}{N})T,\mathbf{x}) = u(\frac{N+1-k}{N}T,\mathbf{x}),
\end{cases} \tag{6.20}
$$

for $k\in [\![1,N]\!]$ with $N>0$ big enough. We take advantage of the stability in $L^\infty(C^{2+\gamma}_{b,\mathbf{d}})$ of Schauder estimates.

We finally conclude by a compactness argument [*] and by uniqueness of the associated martingale problem, cf. [CdRM17]. We have then showed that the *mild* solution satisfies some Schauder estimates. Besov duality property yields that the *mild* solution is also a weak solution.

[*] Schauder estimates (6.4) being independent of the mollification index, we can use the Arzelà–Ascoli theorem.

Our approach cannot allow us to consider Kolmogorov equation without terminal condition as the constants diverges when $T \to +\infty$, namely with the number of iterations of the analysis of the Cauchy problem (6.20). Moreover the constants diverge also with the dimension d[†].

7 Degenerate chain

7.1 Presentation of the model

The kinetic case studied in the previous section is a sub-model of the degenerate chain. The latter is for instance a typical model in seismology. We consider the propagation of a random shaking on several structures which transmit the noise. This model is usually represented by a system of springs attached to each other.

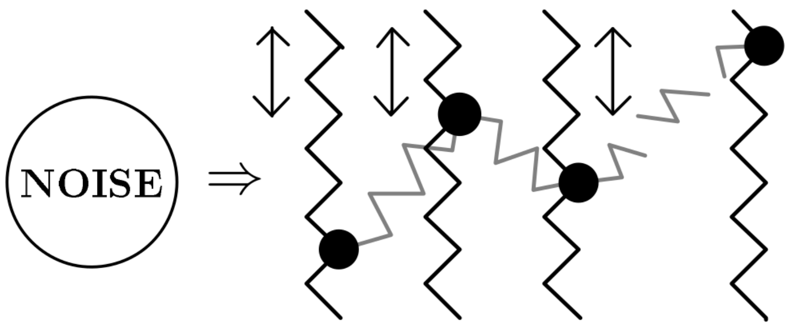


Figure 1: Picture by Delarue Menozzi [DM10].

These degenerate systems are also used for microscopic models associated with the heat diffusion (e.g. Heckmann and Hairer [EH00]).

The stochastic dynamic of this model is:

$$\begin{aligned} d\mathbf{X}_t^1 &= \mathbf{F}_1(t, \mathbf{X}_t^1, \ldots, \mathbf{X}_t^n)dt + \sigma(t, \mathbf{X}_t^1, \ldots, \mathbf{X}_t^n)dW_t, \\ d\mathbf{X}_t^2 &= \mathbf{F}_2(t, \mathbf{X}_t^1, \ldots, \mathbf{X}_t^n)dt, \\ d\mathbf{X}_t^3 &= \mathbf{F}_3(t, \mathbf{X}_t^2, \ldots, \mathbf{X}_t^n)dt, \qquad\qquad t \geqslant 0, \\ &\vdots \\ d\mathbf{X}_t^n &= \mathbf{F}_n(t, \mathbf{X}_t^{n-1}, \mathbf{X}_t^n)dt, \end{aligned} \tag{7.1}$$

with $n \in \mathbb{N}^*$, $\forall i \in [\![2, n]\!]$, $\forall (t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^{nd}$, $D_{\mathbf{x}_{i-1}}\mathbf{F}_i(t, \mathbf{x}_{i-1}, \ldots, \mathbf{x}_n) \in GL_d(\mathbb{R})$. In other words, we consider the weak Hörmander's condition associated with the chain.

[†]The Schauder estimates associated with a parabolic equation with constant coefficients, and with sharp constants not depending on the dimension were established by Krylov and Priola [KP17].

7.2 Schauder estimates

The PDE associated with SDE (7.1) write $\forall T > 0,\ (t, \mathbf{x}) \in [0, T] \times \mathbb{R}^{nd}$,

$$\begin{cases} \partial_t u(t, \mathbf{x}) + \langle \mathbf{F}(t, \mathbf{x}), \mathbf{D}u(t, \mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big(D^2_{\mathbf{x}_1} u(t, \mathbf{x}) a(t, \mathbf{x})\big) = -f(t, \mathbf{x}), \\ u(T, \mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in \mathbb{R}^{nd}, \end{cases} \tag{7.2}$$

where the drift $\mathbf{F}(t, \mathbf{x}) := (\mathbf{F}_1(t, \mathbf{x}), \cdots, \mathbf{F}_n(t, \mathbf{x}))$ corresponds to SDE (7.1), i.e.

$$\forall i \in [\![2, n]\!],\ \mathbf{F}_i(t, \mathbf{x}) := \mathbf{F}_i(t, \mathbf{x}^{i-1:n}),\ \mathbf{x}^{i-1:n} := (\mathbf{x}_{i-1}, \cdots, \mathbf{x}_n), \tag{7.3}$$

The linear case was handled by Lunardi [Lun97], $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{pmatrix} * & \cdots & \cdots & \cdots & * \\ \mathbf{a}_{2,1} & * & \cdots & \cdots & * \\ \mathbf{0}_{d,d} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}_{d,d} & \cdots & \mathbf{0}_{d,d} & \mathbf{a}_{n,n-1} & * \end{pmatrix}, \tag{7.4}$$

and $(\mathbf{a}_{i,j})_{ij \in [\![1,n]\!]^2} \in \mathbb{R}^d \otimes \mathbb{R}^d$ s.t. $\big(\mathbf{a}_{i,i-1}\big)$ are non-degenerate (Hörmander conditions are satisfied).

If σ is constant then $\mathbf{X}_t$ is a Gaussian processes where the covariance matrix $\mathbf{K}_t$ satisfies the "good scaling" property

$$\mathbf{K}_t^{1/2} \asymp t^{-1/2}\mathbb{T}_t := \begin{pmatrix} t^{1/2}\mathbf{I}_{d,d} & \mathbf{0}_{d,d} & \cdots & \mathbf{0}_{d,d} \\ \mathbf{0}_{d,d} & t^{3/2}\mathbf{I}_{d,d} & \mathbf{0}_{d,d} & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{d,d} & \cdots & \mathbf{0}_{d,d} & t^{(2n-1)/2}\mathbf{I}_{d,d} \end{pmatrix}. \tag{7.5}$$

Like for the kinetic version (6.11), each line of the diagonal of the covariance matrix is homogeneous to the successive integrals of the Brownian motion.

Let us define the Hölder norms adapted to this problem:

$$\|u\|_{L^\infty(C^{2+\gamma}_{b,\mathbf{d}})} := \|u\|_{L^\infty} + \|D_{\mathbf{x}_1} u\|_{L^\infty} + \|D^2_{\mathbf{x}_1} u\|_{L^\infty} + \sum_{i=1}^n [D^2_{\mathbf{x}_1} u_i]^\gamma_{\mathbf{d}} + \sum_{i=2}^n [u_i]^{2+\gamma}_{\mathbf{d}},$$

where $[u_i]^{2+\gamma}_{\mathbf{d}}$ are the standard Hölder modulus according to the suitable scaling associated with the variable $\mathbf{x}_i$. In other words, for any $(t, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n) \in [0, T] \times \mathbb{R}^{(n-1)d}$, $[u_i]^{2+\gamma}_{\mathbf{d}}$ is the $\frac{2+\gamma}{2i-1}$-Hölder modulus of $\mathbf{x}_i \mapsto u(t, \mathbf{x})$. In particular, if $i = 1$ then $\frac{2+\gamma}{2i-1} > 2$, hence the required control of the norms $\|D_{\mathbf{x}_1} u\|_{L^\infty}$, $\|D^2_{\mathbf{x}_1} u\|_{L^\infty}$, $\big([D^2_{\mathbf{x}_1} u_i]^\gamma_{\mathbf{d}}\big)_{i \in [\![1,n]\!]}$.

We say that $u \in L^\infty(C^{2+\gamma}_{b,\mathbf{d}})$ if $\|u\|_{L^\infty(C^{2+\gamma}_{b,\mathbf{d}})} < +\infty$. We adapt these norms to the anisotropic Hölder space $L^\infty(C^\gamma_{b,\mathbf{d}})$ where there is derivative in $\mathbf{x}_1$ to consider, and to $C^{2+\gamma}_{b,\mathbf{d}}$ where there is uniform norm in time.

With this notations, Lunardi [Lun97] established the following result: if there is a matrix σ_∞ U.E. such that $\lim_{|\mathbf{x}| \to \infty} \sigma(\mathbf{x}) = \sigma_\infty$, if $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ defined in (7.4), and if $(f, g) \in$

$L^\infty(C^\gamma_{b,\mathbf{d}}) \times C^{2+\gamma}_{b,\mathbf{d}}$ then there a unique *mild* and weak solution $u \in L^\infty(C^{2+\gamma}_{b,\mathbf{d}})$ to the Cauchy problem (7.2) such that:

$$\|u\|_{L^\infty(C^{2+\gamma}_{b,\mathbf{d}})} \leqslant C(\|f\|_{L^\infty(C^\gamma_{b,\mathbf{d}})} + \|g\|_{C^{2+\gamma}_{b,\mathbf{d}}}). \tag{7.6}$$

In 2009, Priola [Pri09] extended this result for $\mathbf{F}_1$ non-linear and $\frac{\gamma}{2i-1}$-Hölder continuous for the variable $\mathbf{x}_i$, $i \in [\![1, n]\!]$.

We obtain a generalisation for drift fully non-linear and without the constraint on the limit of σ.

Theorem 1 (Schauder estimates for the degenerate chain) *If $a = \sigma\sigma^*$ is U.E., $a \in L^\infty(C^\gamma_{b,\mathbf{d}})$, $\mathbf{F}_1 \in L^\infty(C^\gamma_{\mathbf{d}})$, and for any $i \in [\![2, n]\!]$, $\mathbf{F}_i \in L^\infty(C^{2i-3+\gamma}_{\mathbf{d}})$ in particular $\mathbf{F}_i(t, \cdot, \mathbf{x}^{i:n}) \in C^{1+\frac{\gamma}{2(i-1)-1}}$, and if $(f, g) \in L^\infty(C^\gamma_{b,\mathbf{d}}) \times C^{2+\gamma}_{b,\mathbf{d}}$ then there is a unique mild and weak solution $u \in L^\infty(C^{2+\gamma}_{b,\mathbf{d}})$ to the Cauchy problem* (7.2) *such that:*

$$\|u\|_{L^\infty(C^{2+\gamma}_{b,\mathbf{d}})} \leqslant C(\|f\|_{L^\infty(C^\gamma_{b,\mathbf{d}})} + \|g\|_{C^{2+\gamma}_{b,\mathbf{d}}}).$$

The proof in the general case, is similar to the pertubative method showed in the previous section. The main difference in this case is that we need to balance very sharply the different indexes

As said previously, these regularity thresholds are also the optimal ones for the weak uniqueness of the associated SDE, cf. [CdRM17]. We also see these thresholds with the previous heuristic analysis with the Peano's example. Indeed, let us consider the line number i of SDE (7.1) and we look at the variable number $j > i$ of the drift $\mathbf{F}_i$. We suppose that $\mathbf{F}_i$ is β_i^j-Hölder continuous in this variable. We have to compare, at the critical time t_Σ, the contribution of the noise through $\mathbf{x}_{i-1}$, the variable which transmits the noise, with the value of the maximal solution associated with ODE (5.1) for the variable $\mathbf{x}_j$ also by taking account the transmission of the noise to the variable j. For instance for $i = j$, one gets:

$$t_\Sigma^{\frac{1}{1-\beta_i^i}} < t_\Sigma^{\frac{2i-1}{2}} \Leftrightarrow 1 - \beta_i^i < \frac{2}{2i-1} \Leftrightarrow \frac{2i-3}{2i-1} < \beta_i^{i*}. \tag{7.7}$$

8 Strong uniqueness

Our pertubative approach allows us to deal with the problem of strong uniqueness of SDE (7.1), which then generalises the kinetic case (6.1) considered by Chaudru de Raynal [CdR17].

Let us recall that weak uniqueness corresponds to uniqueness in law, and the strong uniqueness corresponds to the uniqueness of the path of a process adapted to the filtration of the noise. In particular strong uniqueness yields weak uniqueness.

The strong uniqueness of the solution of stochastic equations is important in several fields such as in biology and in physics:

- in neuroscience, with neural circuit, cf. [FL16],
- in fluid mechanics, cf. [Fla11] also for a general introduction of the regularisation of the noise.

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*For more details see [CdRM17].

To prove strong uniqueness in the Hölderian case, we used Zvonkin Veretennikov transform, cf. [Zvo74] and [Ver83], which strongly relies on the study of Kolmogorov equation.

Indeed, if all the coefficients σ and $\mathbf{F}$ are Lipschitz continuous, the strong uniqueness is obvious. Let us take two solutions $\mathbf{X}_t, \mathbf{X}'_t$ of SDE (7.1), we readily write:

$$\begin{aligned}
\mathbb{E}\big|\mathbf{X}_t - \mathbf{X}'_t\big|^2 &= \mathbb{E}\Big|\int_0^t \mathbf{F}(\mathbf{X}_s) - \mathbf{F}(\mathbf{X}'_s)ds + \int_0^t \sigma(\mathbf{X}_s) - \sigma(\mathbf{X}'_s)dW_s\Big|^2 \\
&\leqslant 2[\mathbf{F}]_1\mathbb{E}\Big|\int_0^t |\mathbf{X}_s - \mathbf{X}'_s|ds\Big|^2 + 2[\sigma]_1\int_0^t \mathbb{E}\big|\mathbf{X}_s - \mathbf{X}'_s\big|^2 ds \\
&\overset{\text{Cauchy-Schwarz}}{\leqslant} 2\big([\mathbf{F}]_1 t + [\sigma]_1\big)\int_0^t \mathbb{E}\big|\mathbf{X}_s - \mathbf{X}'_s\big|^2 ds.
\end{aligned} \tag{8.1}$$

By Grönwall's inequality we directly get $(\mathbf{X}_t)_{t\geqslant 0} = (\mathbf{X}'_t)_{t\geqslant 0}$ almost surely. The path uniqueness is then established and Yamada Watanabe theorem [YW71] allows us to conclude thanks to the weak existence, cf. [CdRM17].

The Hölderian case is much more delicate to handle.

Theorem 2 (Strong uniqueness for the degenerate chain and Hölder continuous drift) *If for each $i \in [\![1,n]\!]$, $[(\mathbf{F}_i)_j]_{\beta_j}{}^{\dagger} < \infty$ with $\beta_j \in (\frac{2j-2}{2j-1}, 1]$ and for $i \geqslant 2$, $[(D_{\mathbf{x}_{i-1}}\mathbf{F}_i)]_\eta < \infty$, $\eta > 0$ "small", and σ Lipschitz continuous then there is a unique strong solution to SDE* (7.1).

Let us remark that these thresholds on the drift regularity are stronger than for Theorem 1 and for weak uniqueness. They are sharp for this method, but we do not know, today, if there is counterexample: if there is a SDE with a drift $\mathbf{F}$ more smooth than in Theorem 1 but less than in Theorem2 where strong uniqueness of the solution fails to be true.

Catelier and Gubinelli [CG16] find the same thresholds for a fractional Brownian motion with self-similarity index corresponding to the iterated integrals of the Brownian motion. Their results may suggest that under our threshold strong uniqueness is not satisfied.

Idea of the proof

Like in the Lipschitz case, we show the uniqueness pathwise and we use Yamada Watanabe theorem and the weak existence to establish strong uniqueness.

Similarly to the proof of Schauder estimates the starting point is the mollification of the coefficients of SDE (7.1) to be sure that each used function in the analysis exists and is uniquely defined. Again, we omit the mollification index. We consider the Cauchy problem associated with (7.1):

$$\begin{cases} \partial_t u(t,\mathbf{x}) + \langle \mathbf{F}(t,\mathbf{x}), \mathbf{D}u(t,\mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big(D^2_{\mathbf{x}_1}u(t,\mathbf{x})a(t,\mathbf{x})\big) = -\mathbf{F}(t,\mathbf{x}),\ t \in [0,T), \\ u(T,\mathbf{x}) = 0,\ \mathbf{x} = (\mathbf{x}_1,\cdots,\mathbf{x}_n) \in \mathbb{R}^{nd}. \end{cases} \tag{8.2}$$

Thanks to the Itô's formula and to (8.2) we write:

$$\int_0^t \mathbf{F}(s,\mathbf{X}_s)ds = -u(0,\mathbf{x}) + u(t,\mathbf{X}_t) - \int_0^t \mathbf{D}u(s,\mathbf{X}_s)B\sigma(s,\mathbf{X}_s)dW_s. \tag{8.3}$$

†This is the Hölder modulus of $\mathbf{F}_i$ for the variable $\mathbf{x}_j$.

The main idea of Zvonkin Veretennikov transform consists in taking advantage of the "parabolic bootstrap" for the solution u of Kolmogorov equation (8.2) in order to show, somehow, that the r.h.s. in (8.3) is more smooth than the drift in the l.h.s.. This phenomenon is also linked with the averaging of the Brownian process, cf. [Dav07], [CG16].

The trick in (8.3) is to write the dynamic of $\mathbf{X}_t$ without the drift. Indeed, we first write:

$$\mathbb{E}\Big[\sup_{s\leqslant t}|\mathbf{X}_s - \mathbf{X}'_s|^2\Big] = \mathbb{E}\Big[\sup_{s\leqslant t}|\{\mathbf{X}_s - u(s,\mathbf{X}_s)\} + u(\mathbf{X}_s) - s(\mathbf{X}'_s)\} + u(\mathbf{X}'_s) - \mathbf{X}'_s\}|^2\Big]. \tag{8.4}$$

The first and the last terms are handled thanks to (8.3), namely

$$\mathbf{X}_s - u(s,\mathbf{X}_s) = \int_0^s \sigma(v,\mathbf{X}_v)dW_v - u(0,\mathbf{x}) - \int_0^s D_{\mathbf{x}_1}u(v,\mathbf{X}_v)\sigma(v,\mathbf{X}_v)dW_v. \tag{8.5}$$

The second contribution in the l.h.s. in (8.4) is dealt by the regularity of the function u. The backbone of the analysis consists in showing that there is a constant $C_T > 0$ such that $\lim_{T\to 0} C_T \to 0$ and

$$\|\mathbf{D}u\|_\infty + \|\mathbf{D}(D_{\mathbf{x}_1}u)\|_\infty \leqslant C_T, \tag{8.6}$$

where $\mathbf{D}$ is the full gradient. We need the control of the crossed derivatives $\mathbf{D}(D_1 u)$ because of the last term which appears in (8.5). To establish this control of derivatives, we adapt the pertubative method (cf. Section 6) to this framework, except that now we have to control the full gradient and the crossed derivatives of u. Because the source function in Kolmogorov equation is the drift $\mathbf{F}$, some new constraints on the smoothness on $\mathbf{F}$ are required.

After proving identity (8.6) we obtain

$$\mathbb{E}[\sup_{0\leqslant s\leqslant T}\big|\mathbf{X}_s - \mathbf{X}'_s\big|^2] \leqslant C_T\mathbb{E}[\sup_{0\leqslant s\leqslant T}|\mathbf{X}_s - \mathbf{X}'_s|^2].$$

We conclude on the strong uniqueness in the interval $[0,T]$, for T small enough by a circular argument. We repeat this argument to be able to consider any finite final time T. $\square$

To conclude this summary of the second part of my thesis, I would like to say that the phenomenon of regularisation by a noise is a deep and rich research topic. There is still a lot to understand, such as the interpretation of the thresholds for the strong uniqueness. In the weak case, even non-degenerate, the Peano's example suggests us that the minimal regularity of $\mathbf{F}_1$ should be $C^{-1} = B^{-1}_{\infty,\infty}$. It seems that a rough path approach yields to the constraint $\mathbf{F}_1 \in B^{-1+\gamma}_{\infty,\infty}$ with $\gamma > 1/3$, cf. [DD16] and [CC18]. For the strong uniqueness, Bass and Chen [BC01] (see also [Bar82]) found a counterexample for $\gamma < 1/2$. I would like to keep investigating these questions about minimal regularity in Besov and Triebel–Lizorkin spaces.

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