Vlasov-Fokker-Planck with general potentials

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Steady states of VFP

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Introduction

Self-consistent Vlasov-Fokker-Planck

Consider a system of particles $\mathbb{R}^d \times \mathbb{R}^d$, described at time $t \geq 0$ by its phase-space distribution function $F(t, x, v)$, satisfying

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nabla_v \cdot (vF + \nabla_v F)$$

- Random fluctuations and damping of the velocity (Fokker-Planck)
- Particles localized in a region of space by an outside force $\nabla_x V$
- Particle at y affects particle at x with a force $\nabla_x k(x - y)$

$$\Psi_F(x) = \int_{\mathbb{R}^{2d}} k(x - y) \rho_F(y) \mathrm{d}y \,, \qquad \rho_F(x) = \int_{\mathbb{R}^d} F(x, v) \mathrm{d}v \,.$$

Self-consistent Vlasov-Fokker-Planck

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Why this equation is interesting/hard at first glance:

- Degeneracy: diffusion in v only and vanishes on $\mathcal{G}(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$
- Non-linearity is non-local

A less obvious reason:

- Phase transition in the strongly non-linear (large mass) regime

$$\partial_t G + v \cdot \nabla_x G - \nabla_x (M\Psi_G + V) \cdot \nabla_v G = \nabla_v \cdot (vG + \nabla_v G)$$

where $M = \int_{\mathbb{R}^{2d}} F(t)\mathrm{d}x\mathrm{d}v$ is the (conserved) mass and $G = F/M$

Interaction potential

Symmetric and skew-symmetric parts of the convolution operator:

$$\mathcal{K}\rho = \int_{\mathbb{R}^d} k(x-y)\rho(y)\,\mathrm{d}y\,, \qquad \mathcal{K}^\alpha\rho = \int_{\mathbb{R}^d} k^\alpha(x-y)\rho(y)\,\mathrm{d}y$$

associated with the even and odd parts of the kernel k:

$$k^{\mathrm{e}}(x) = \frac{1}{2}(k(x)+k(-x))\,, \qquad k^{\mathrm{o}}(x) = \frac{1}{2}(k(x)-k(-x))\,.$$

“Ideal” example we have in mind

- In plasma physics: $\mathcal{K}$ is symmetric and positive

$$\text{(Coulomb)} \qquad k_C(x) = \frac{I}{|x|}, \quad d=3, \quad I>0.$$

“Bad” examples we have in mind

- In particle accelerator physics: $\mathcal{K}$ is non-symmetric and $k \in W^{1,\infty}$:
- Kuramoto $k = -\cos(\omega x)$: $\mathcal{K}$ is symmetric but negative

Positive symmetric potentials: example of a plasma

Consider the Vlasov-Poisson-Fokker-Planck in dimension 3

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nabla_v \cdot (vF + \nabla_v F) \,, \\ -\lambda^2 \Delta \Psi_F(t,x) = \rho_F, \\ F|_{t=0} = F_{\text{in}}. \end{cases}$$

corresponding to the Coulomb potential $k_C(x) = \frac{I}{|x|}$ with $I = C\lambda^{-2}$.

Theorem (Bouchut, Dolbeault '95 : unconditional cvg)

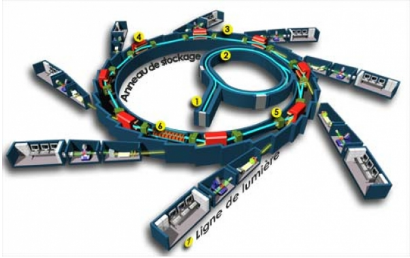
Assume that F_{in} satisfies physical bounds (mass, entropy, total energy) and $\nabla \Psi_F \in L^\infty_{t\,loc} L^\infty_x$, then

$$F(t) \xrightarrow{t\to\infty} F_\star \quad in \quad L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)\,,$$

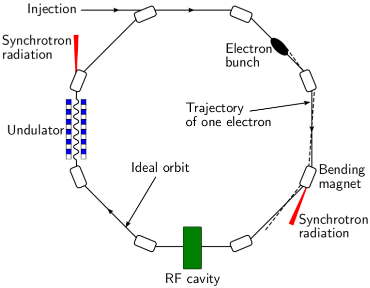
where $F_\star$ is the unique steady state.

First quantitative results by [Hérau, Thomann '16] in weakly nonlinear regime ($\lambda \gg 1$)

Asymmetric potentials: example of a particle accelerator

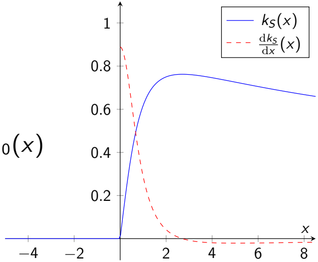


[Source: synchrotron-soleil.fr]



[Roussel, PhD thesis, '14]

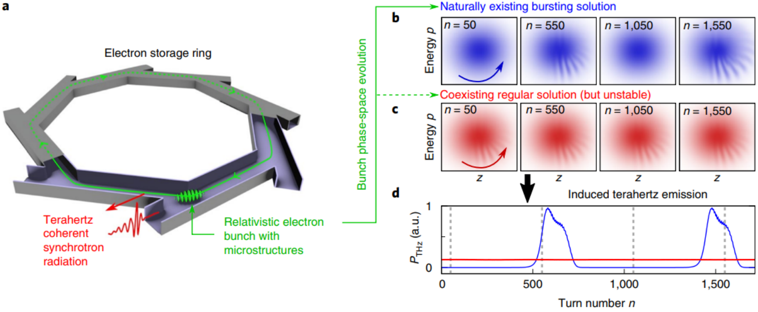
$$k_S(x) = 2I \frac{\cosh\left[\frac{5}{3}\sinh^{-1} x\right] - \cosh\left[\sinh^{-1} x\right]}{\sinh\left[2\sinh^{-1} x\right]} \mathbb{1}_{x>0}(x)$$



Asymmetric potentials: micro-bunching instabilities

Numerical evidences and linear stability analysis shows that

- At low currents $I \ll 1$ (weakly nonlinear), there is a unique asymptotically stable steady state.
- At high currents $I \gg 1$ (strongly nonlinear), dynamics is more complex



[Evain et. al., Nature Physics '19]

Motivations

Characterize steady states and asymptotic stability regimes for

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \left(\mathcal{K} \rho_F + V \right) \cdot \nabla_v F = \nu \nabla_v \cdot \left(v F + \nabla_v F \right),$$

with as general as possible $\mathcal{K}$ and V, and quantitatively.

What we want to address:

- Strongly nonlinear regime; $\mathcal{K}$ large (at least in some sense)
- Consideration of asymmetric interaction kernels

Steady states of VFP

Gibbs steady states

For symmetry reasons any reasonable solution to

$$v \cdot \nabla_x F_\star - \nabla_x \left(\Psi_{F_\star} + V\right) \cdot \nabla_v F_\star = \nu \nabla_v \cdot \left(v F_\star + \nabla_v F_\star\right)$$

with $\Psi_{F_\star} = \mathcal{K}\rho_{F_\star}$ cancels both sides:

- Vlasov: $v \cdot \nabla_x F_\star - \nabla_x \left(\Psi_{F_\star} + V\right) \cdot \nabla_v F_\star = 0$ (“odd” wrt v)
- Fokker-Planck: $\nabla_v \cdot \left(v F_\star + \nabla_v F_\star\right) = 0$ (“even” wrt v)

Therefore

$$F_\star(x, v) = \rho_\star(x)\mathcal{G}(v)$$

with $\rho_\star$ is a solution of the

Gibbs fixed point problem

$$\rho_\star = \mathcal{T}(\rho_\star) := \frac{\mathcal{S}(\rho_\star)}{\|\mathcal{S}(\rho_\star)\|_{L^1}}, \qquad \mathcal{S}(\rho) := e^{-V - \mathcal{K}\rho}\,.$$

a.k.a. Haissinski eq. in particle accelerator community [Haissinski '73].

Assumptions on potentials

Assume that the confinement potential V satisfies, for some $N \geq 2$

$$\forall n \leq N, \quad |\nabla^n V|^{\frac{N}{n}} e^{-V} \in L^1 \cap L^\infty \,, \qquad \int_{\mathbb{R}^d} e^{-V} \mathrm{d}x = 1$$

Assume $\mathcal{K}$ is bounded for some $p, q \in [2, \infty]$ and monotonous:

$$\mathcal{K} : L^1 \cap L^2 \to L^p \,, \qquad \nabla\mathcal{K} : L^1 \cap L^2 \to L^q$$

$$\rho \geq 0 \Rightarrow \mathcal{K}\rho \geq 0$$

Assume the following behavior of $\mathcal{K}$ against the confinement profile e^{-V}:

$$\mathcal{K}^* \left(e^{-V}\right) \in L^\infty$$

Well-posedness of the steady state problem

Theorem (G, Herda. '24)

The stationary solutions to Vlasov-Fokker-Planck equation are of the form

$$F_\star(x,v) = \rho_\star(x) M(v), \quad \rho_\star = e^{-V_\star},$$

with $\rho_\star$ a solution of the Gibbs fixed point problem.

- *Under the previous hypotheses, there exists at least one solution.*
- *If we assume for some $0 < \underline{\kappa}^e \ll 1$ and any zero-mean $h \in L^1 \cap L^2$*

  $$\langle \mathcal{K} h, h \rangle = \langle \mathcal{K}^e h, h \rangle \geq -\underline{\kappa}^e \|h\|^2_{L^1 \cap L^2},$$

  then the steady state is unique.

See also [Carrillo, Gvalani, Pavliotis, Schlichting '20], [Cesbron, Herda '24].

Uniqueness

Let us introduce the free energy functional

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^d} \left(V + \frac{1}{2}\mathcal{K}^e(\rho) \right) \rho \, \mathrm{d}x + \int_{\mathbb{R}^d} \rho \log \rho \, \mathrm{d}x$$

If $\mathcal{K}^o \neq 0$, the classical argument no longer works:

$$\rho \text{ fixed point } \not\Leftrightarrow \rho \text{ critical point of } \mathcal{F}$$

Indeed, for all zero mean h

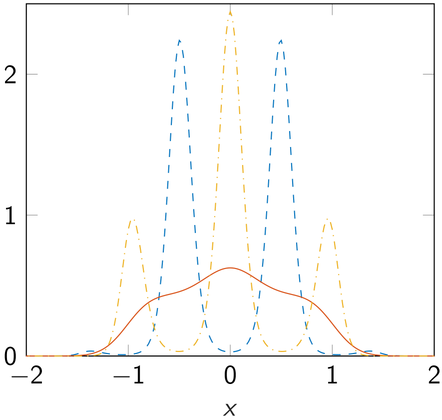
$$\rho \text{ fixed point } \Rightarrow \mathrm{d}_\rho \mathcal{F}[\rho] \cdot h = - \int_{\mathbb{R}^d} \mathcal{K}^o(\rho) h \, \mathrm{d}x$$

But it still holds that strict convexity of $\mathcal{F} \Rightarrow$ uniqueness for $\rho = \mathcal{T}(\rho)$:

$$\rho_0, \, \rho_1 \text{ two fixed points} \Longrightarrow (\mathrm{d}_\rho \mathcal{F}[\rho_1] - \mathrm{d}_\rho \mathcal{F}[\rho_0]).(\rho_1 - \rho_0) = 0$$

therefore $\rho_0 = \rho_1$

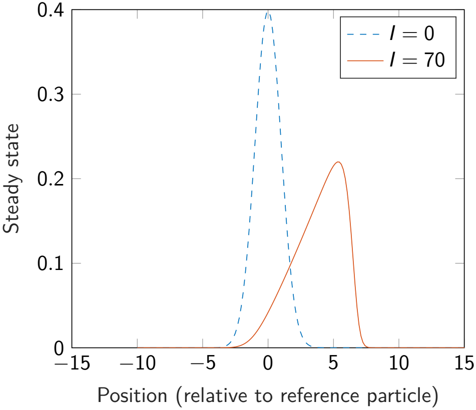
A case of non-uniqueness: Kuramoto $k(x) = -I\cos(2\pi x)$



Three steady states for $I = \underline{\kappa}^e = 3$

A case of non-symmetric potential

Interaction $k(x) = Ik_S(x)$



Algo for numerical resolution [Warnock, Bane '18].

Quantitative stability

Assumptions on the confining potential

We make the following integrability and boundedness conditions:

$$\left(1+|\nabla V|^2\right) e^{-V} \in L^1 \cap L^\infty \quad \text{and} \quad \int_{\mathbb{R}^d} e^{-V(x)} \mathrm{d}x = 1\,.$$

Moreover, in order that the hierarchy

$$\left\{H^s\left(\mathbb{R}^d, e^{-V} \mathrm{d}x\right)\right\}_{0 \le s \le 1} \quad \text{"behaves" like} \quad \left\{H^s\left(\mathbb{T}^d, \mathrm{d}x\right)\right\}_{0 \le s \le 1}$$

we also assume that for any $\varepsilon > 0$

$$\forall x \in \mathbb{R}^d, \quad |\nabla^2 V(x)| \le \varepsilon |\nabla V(x)| + C_\varepsilon\,,$$

and the measure $e^{-V} \mathrm{d}x$ admits a Poincaré inequality

$$\int_{\mathbb{R}^d} |u|^2 e^{-V} \mathrm{d}x - \left(\int_{\mathbb{R}^d} u e^{-V} \mathrm{d}x\right)^2 \le C_P \int_{\mathbb{R}^d} |\nabla_x u|^2 e^{-V} \mathrm{d}x\,.$$

Assumptions on the interaction potential

For some $p, q \in [2, \infty]$ with $q > d$, one has the regularity estimates

$$\|\mathcal{K}^\alpha \rho\|_{L^p} + \|\nabla \mathcal{K}^\alpha \rho\|_{L^q} \le \overline{\kappa}^\alpha \|\rho\|_{L^1 \cap L^2}, \qquad \alpha = e, o\,.$$

There is $\underline{\kappa}^{\mathrm{e}} > 0$ such that one has the bound from below

$$\langle \mathcal{K}^{\mathrm{e}} h, h \rangle \ge -\underline{\kappa}^{\mathrm{e}} \|h\|^2_{L^1 \cap L^2}, \quad \text{for all } h \in L^1 \cap L^2 \text{ s.t. } \int h = 0.$$

Finally we assume the monotonicity property

$$\rho \ge 0 \Rightarrow \mathcal{K}\rho \ge 0, \text{ for all } \rho \in L^1 \cap L^2.$$

Quantitative local asymptotic stability

Theorem (G, Herda. '24)

There are constants $\delta^e > 0$ *and* $\delta^o > 0$ *such that if*

$$\underline{\kappa}^e < \delta^e(\theta, \overline{\kappa}_{max}, R_V) \qquad \text{and} \qquad \overline{\kappa}^o < \delta^o(\overline{\kappa}_{max}, R_V, \underline{\kappa}^e, \nu),$$

the unique steady state of VFP is stable in the following sense. For any $s \in [0,1]$ *such that*

$$s > s_c := \frac{3}{2}\left(\frac{d}{q} - \frac{1}{3}\right),$$

there is a constant $R > 0$ *such that if*

$$\|F_{in} - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} < R,$$

then VFP has a unique solution $F \in \mathcal{C}([0,\infty); H^s_x L^2_v(F_\star^{-1}))$*. Moreover, there are constants* $C > 0$ *and* λ *such that for all* $t > 0$

$$\|F(t) - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} \leq C\|F_{in} - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} e^{-\lambda t}$$

Finally, $F_{in} \mapsto F$ *is Lipschitz continuous.*

Corollary: Vlasov-Poisson-Fokker-Planck $\mathcal{K} = (-\lambda^2\Delta)^{-1}$

Hypotheses on the potential

- Regularity assumptions on $\mathcal{K}$ and $\nabla\mathcal{K}$ are consequences of Hardy-Littlewood-Sobolev / elliptic regularity.
- No smallness for λ because of positivity and symmetry of $-\Delta$:

$$\mathcal{K} = \mathcal{K}^e = \left(-\lambda^2\Delta\right)^{-1} \geq 0 \quad \Rightarrow \quad \overline{\kappa}^o = \underline{\kappa}^e = 0$$

Consequences of our result

- Quantitative decay estimate with constructive constants
- Holds for any Debye length λ (but constants are $\mathcal{O}(\lambda^{-2})$)
- Regularity on initial data ($H^s_x L^2_v$, $s > 1/4$) is lowered compared to former results [Hérau, Thomann '16], [Toshpulatov '23]

Linearized VFP

- Perturbative setting around Gibbs steady state

$$F = F_\star(1+f)$$

- Hilbertian setting

$$\|f\|^2 = \iint_{\mathbb{R}^{2d}} f^2 F_\star \mathrm{d}x\mathrm{d}v$$

and ∇_v^* and ∇_x^* adjoints for the corresponding scalar product.

- The VFP equation rewrites in linearized form

$$(\partial_t + \Lambda)f(t) + v\cdot\nabla_x \mathcal{K}\rho_f(t) = \nabla_v^*\varphi[f], \qquad f|_{t=0} = f_{\mathrm{in}}\,.$$

with the linear part

$$\Lambda := \nu\nabla_v^*\nabla_v + v\cdot\nabla_x - \nabla_x V_\star\cdot\nabla_v =: \Lambda^{\mathrm{sym}} + \Lambda^{\mathrm{skew}}$$

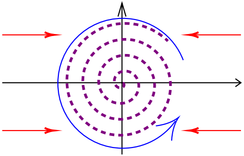
and the nonlinear part

$$\varphi[f] = f\nabla_x\mathcal{K}\rho_f \quad \text{and} \quad \mathcal{K}\rho_f = \iint k(\cdot - y) f(y,w) F_\star(y,w)\mathrm{d}y\mathrm{d}w$$

Hypocoercivity in a nutshell: a 2D toy model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} y$$

$$\text{EigenV} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$



- Decay $y(t) = \mathcal{O}\left(e^{-t/2}\right)$ can't be deduced from the energy estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|^2 = -y_2^2(t)$$

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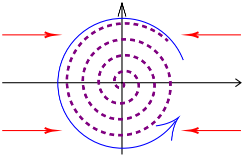

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- By introducing the equivalent (squared) norm ($|\eta| < 1$)

$$H(y) = y_1^2 + y_2^2 + 2\eta y_1 y_2$$

one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H(y(t)) \leq -\lambda_\eta H(y(t))$$

This strategy also works in infinite dimension!

L^2 hypocoercivity

- Λ is not coercive for the canonical norm:

$$\langle \Lambda f, f \rangle = \nu \|\nabla_v f\|^2 \gtrsim \|f - \Pi f\|^2 , \qquad \Pi f = \int f \mathcal{G} \, \mathrm{d}v$$

We wish to add a cross term $\langle Af, f \rangle$ making it coercive:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Af, f \rangle = \left\langle A \Lambda^{\mathsf{skew}} f, f \right\rangle + \cdots \gtrsim \|\Pi f\|^2 + \mathcal{O}\left(\|f - \Pi f\|^2 \right)$$

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- Solution from [Dolbeault, Mouhot, Schmeiser '15]:

$$A := \left(\mathrm{Id} + \left(\Lambda^{\mathsf{skew}} \Pi\right)^* \Lambda^{\mathsf{skew}} \Pi\right)^{-1} \left(\Lambda^{\mathsf{skew}} \Pi\right)^*, \qquad \|\Lambda^{\mathsf{skew}} \Pi\| \geq c \|\Pi f\|$$

which is chosen so that

$$A \Lambda^{\mathsf{skew}} \Pi = \frac{|\Lambda^{\mathsf{skew}} \Pi|^2}{1 + |\Lambda^{\mathsf{skew}} \Pi|^2} \geq \frac{c^2}{1 + c^2} \Pi$$

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- Λ is coercive for the equivalent norm $\|f\|^2 + \eta \langle A f, f \rangle$

The linearized free energy norm

- If the linear interaction term is viewed as a perturbation of Λ

$$(\partial_t + \Lambda) f(t) + v \cdot \nabla_x \mathcal{K} \rho_f(t) = \nabla_v^* \varphi(t)$$

asymptotic stability will be obtained only for $\|\mathcal{K}\| \ll 1$!

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- Change Hilbert geometry as in [Addala, Dolbeault, Li, Tayeb, '23]:
  - When $\mathcal{K}$ is symmetric, kinetic free energy=Lyapunov functional:

$$\mathcal{F}^{\text{kin}}[F] := \int_{\mathbb{R}^{2d}} F \log F \, \mathrm{d}x\mathrm{d}v + \int_{\mathbb{R}^d} \rho_F \left(\frac{1}{2} \mathcal{K}^e \rho_F + V \right) \mathrm{d}x\mathrm{d}v$$

$\Rightarrow$ its linearization provides a "natural" norm:

$$|||f|||^2 := \int_{\mathbb{R}^{2d}} f^2 F_\star \mathrm{d}x\mathrm{d}v + \int_{\mathbb{R}^d} (\mathcal{K}^{\mathrm{e}} \rho_f) \rho_f \mathrm{d}x = \mathrm{d}^2_\rho \mathcal{F}^{\text{kin}}[F_\star] \cdot (f)^2$$

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  - New Hilbert structure $\Rightarrow$ new symmetric/skew decomposition:

$$\widetilde{\Lambda} := \nu \nabla_v^* \nabla_v + v \cdot \nabla_x - \nabla_x V_\star \cdot \nabla_v + \boldsymbol{v \cdot \nabla_x \mathcal{K}^e \rho_f} =: \widetilde{\Lambda}^{\text{sym}} + \widetilde{\Lambda}^{\text{skew}}$$

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- Apply L^2–hypocoercivity in this new Hilbert structure to

$$(\partial_t + \widetilde{\Lambda}) f(t) + v \cdot \nabla_x \mathcal{K}^o \rho_f(t) = \nabla_v^* \varphi(t)$$

where the interaction term is a small perturbation of size $\|\mathcal{K}^o\| \ll 1$.

L^2–hypocoercivity on x derivative

We also use the previous L^2–hypocoercivity on the x derivative of VFP:

$$\partial_t (\nabla_x f) + \widetilde{\Lambda} (\nabla_x f) + v \cdot \nabla_x \mathcal{K}^o \rho_{\nabla_x f} = \nabla_v^* (\nabla_x \varphi) + \dots .$$

Interpolation yields hypocoercivity in $H^s_x L^2_v (F_\star \mathrm{d}x \mathrm{d}v)$ with $s \in [0,1]$

Proposition

Assuming $\underline{\kappa}^e < \delta^e$ and $\overline{\kappa}^o < \delta^o$, there exists λ and C such that

$$\sup_{t \geq 0} e^{2\lambda t} \|f(t)\|^2_{H^s_x L^2_v(F_\star)} + \int_0^\infty e^{2\lambda t} \|\nabla_v f(t)\|^2_{H^s_x L^2_v(F_\star)} \mathrm{d}t \leq C \left(\|f_{in}\|^2_{H^s_x L^2_v(F_\star)} + \int_0^\infty e^{2\lambda t} \|\varphi(t)\|^2_{H^s_x L^2_v(F_\star)} \mathrm{d}t \right) .$$

Hypoellipticity: regularizing version of hypocoercivity

Zeroth order estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0(f) + \|\nabla_v f\|^2 \leq 0\,. \tag{1}$$

First order estimates:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2 + c\|\nabla_v^2 f\|^2 \leq \varepsilon\|\nabla_x f\|^2 + C_\varepsilon(\dots)\,. \tag{2}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2 + c\|\nabla_v \nabla_x f\|^2 \leq C\left(\|\nabla_x f\|^2 + \dots\right) \tag{3}$$

The famous cross term estimate [Hérau, Nier], [Villani]...

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_x f, \nabla_v f\rangle + c\|\nabla_x f\|^2 \leq \varepsilon\|\nabla_x \nabla_v f\|^2 + C_\varepsilon(\dots) \tag{4}$$

Combination of (1)–(4) with time weights ⇒ decay/regularization

$$\|\nabla_x f(t)\| \leq Ce^{-\lambda t} t^{-3/2}\|f_{\mathrm{in}}\| \quad \dots$$

Interpolation and nonlinear estimates

We interpolate the previous results on the linear flow $(\varphi, f_{\mathrm{in}}) \mapsto f$

$$\|\varphi\|_{\mathbb{H}^s}^2 = \int_0^\infty e^{2\lambda t} \left(t^{3(1-s)} \|\varphi\|^2_{H^1_x L^2_v(F_\star)} + \dots \right) \mathrm{d}t$$

$$\|f\|^2_{\boldsymbol{\mathcal{X}}^s} := \sup_{t>0} e^{2\lambda t} \left(\|f\|_{H^s_x L^2_v(F_\star)} + t^{3(1-s)} \|f\|^2_{H^1_x L^2_v(F_\star)} + \dots \right) + \int_0^\infty e^{2\lambda t} \left(t^{2(1-s)} \|f\|^2_{H^1_x L^2_v(F_\star)} + \dots \right) \mathrm{d}t$$

Proposition

For any given $s \in [0,1]$,

$$\|f\|_{\boldsymbol{\mathcal{X}}^s} \leq C \left(\|f_{in}\|_{H^s_x L^2_v(F_\star)} + \|\varphi\|_{\mathbb{H}^s} \right).$$

If additionally $s > s_c := \frac{3}{2} \left(\frac{d}{q} - \frac{1}{3} \right)$, *then*

$$\|f \nabla_x \psi_g\|_{\mathbb{H}^s} \leq C \|f\|_{\boldsymbol{\mathcal{X}}^s} \|g\|_{\boldsymbol{\mathcal{X}}^s}.$$

Concluding remarks

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Takeaway message

- About $\mathcal{K}^e$
  - Uniqueness of steady states lost if $\mathcal{K}^e$ is very negative
  - Natural Hilbert norm includes $\mathcal{K}^e$: no upper-bound necessary
- About $\mathcal{K}^o$
  - Stability may be lost if $\mathcal{K}^o$ is large but not uniqueness
- Mixing hypocoercivity techniques (L^2 and H^1) : finer estimates + reduce regularity assumption on initial data

Perspectives

- Phase transitions in strongly nonlinear regime
- McKean-Vlasov limit (diffusive regime)
- Non-perturbative regime (unique steady states is attractive ?)

Thanks for your attention!

[G, Herda., *Well-posedness and long-time behavior for self-consistent Vlasov-Fokker-Planck equations with general potentials*. arXiv:2408.16468]

[Cesbron, Herda, *On a Vlasov-Fokker-Planck equation for stored electron beams*. Journal of Differential Equations, 2024]