Université de Paris

# On the Boltzmann equation with long range interactions

Pierre Gervais

ENS | PSL

# Introduction

## The Boltzmann equation

Assume...

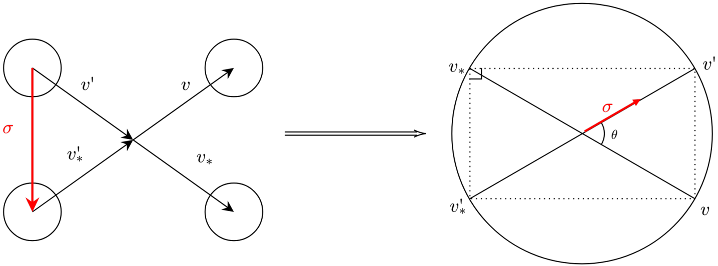
1. Particles travel in **straight lines**
2. **Only pairs** of particles interact
3. **Conservation** of mass / momentum / energy
4. Strength of interaction $=$ (distance)$^{-p}$, with $2 < p < \infty$,

...then the density of particles $F = F(t, x, v)$ at position $x \in \mathbb{R}^3$ traveling at velocity $v \in \mathbb{R}^3$ evolves according to the **Boltzmann equation**:

$$(\partial_t + v \cdot \nabla_x)\, F(t, x, v) = Q\Big(F(t, x), F(t, x)\Big)(v)$$

# Introduction

## Definition of the collision operator

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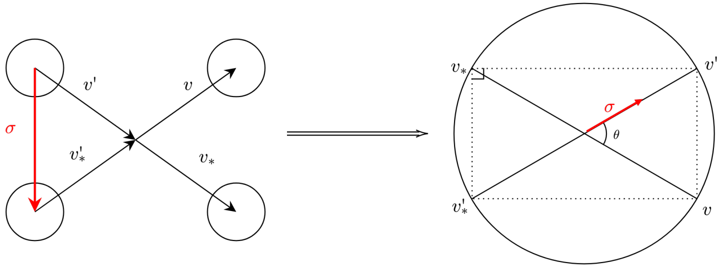


$$v' = v'(v, v_*, \sigma), \quad v'_* = v'(v, v_*, \sigma)$$

$$Q(F,F)(v) = \int_{\mathbb{S}^2_\sigma \times \mathbb{R}^3_{v_*}} |v - v_*|^\gamma b(\cos\theta) \Big[ F(v'_*)F(v') - F(v_*)F(v) \Big] \mathrm{d}\sigma \mathrm{d}v_*$$

$$b(\cos\theta) \approx \theta^{-(2+2s)}$$

$$s = \frac{1}{p-1} \in (0,1), \quad \gamma = \frac{p-5}{p-1} \in (-3,1)$$

# Introduction

## Macroscopic quantities and equilibria

- **Conservation** of mass, momentum, energy :

$$\partial_t \left\{ \int F \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v \right\} + \nabla_x \cdot \left\{ \int F \begin{pmatrix} v \\ v \otimes v \\ v|v|^2 \end{pmatrix} \mathrm{d}v \right\} = \int Q \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = 0$$

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- **Dissipation** of entropy (H-Theorem):

$$\partial_t \left\{ \int F \log F \mathrm{d}v \right\} + \nabla_x \cdot \left\{ \int vF \log F \mathrm{d}v \right\} = \int Q(F,F) \log F \mathrm{d}v \leq 0$$

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... minimized by «maxwellians» :

$$Q(F,F) = 0 \Leftrightarrow \exists (R,U,T),\ F = R \exp\left( -\frac{|v-U|^2}{2T} \right)$$

$\Rightarrow M = \exp\left(-\frac{|v|^2}{2}\right)$ is an equilibrium.

# Perturbation of equi. : space homogen. case

## Linearization of the equation

Consider $F = F(v)$ and linearize around the equi. $F(v) = M(v) + f(v)$:

$$\partial_t f = \mathcal{L} f + Q(f, f)$$

where we defined

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- Microscopic conservation laws + $M(v)M(v_*) = M(v')M(v'_*)$ imply

$$\langle \mathcal{L} f, g \rangle_{L^2(M^{-1} \mathrm{d}v)} = \langle f, \mathcal{L} g \rangle_{L^2(M^{-1} \mathrm{d}v)}$$

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- Null space and the $L^2$-othogonal of the range differ by a factor $M$:

$$\varphi = 1, v, |v|^2 \Longrightarrow \mathcal{L}(\varphi M) = 0, \text{ and } \int \varphi (\mathcal{L} f) \mathrm{d}v = 0$$

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- Linearized $H$-Theorem:

$$0 \geq \int \log(M + f)Q(M + f)\mathrm{d}v = \langle \mathcal{L}f, f\rangle_{L^2(M^{-1}\mathrm{d}v)} + \mathcal{O}(\|f\|^3)$$

**Conclusion:** $= L^2\left(M^{-1}\mathrm{d}v\right)$ is a natural functional space

# Space homog. equation with gaussian weights

## Linearization of the equation with gaussian weights

Weighted linearization $F = M + M^{1/2} f$:

$$\partial_t f = Lf + \Gamma(f, f)$$

- $L$ is self-adjoint in $L^2_v$
- $L$ has a 5-dimensional null-space :

$$N(L) = \left\{ M^{1/2}, v_1 M^{1/2}, v_2 M^{1/2}, v_2 M^{1/2}, |v|^2 M^{1/2} \right\}$$

- $L$ dissipates some quantity $\|f\|^2_{H^{s,*}}$ :

$$f \perp N(L) \Longrightarrow \langle Lf, f \rangle_{L^2_v} \approx -\|f\|^2_{H^{s,*}}$$

$$Lf := M^{-1/2} \left\{ Q(M, M^{1/2} f) + Q(M^{1/2} f, M) \right\},$$

$$\Gamma(f, g) := M^{-1/2} Q(M^{1/2} f, M^{1/2} g)$$

# Space homog. equation with gaussian weights

## Properties of the non-linearity $\Gamma$

$$\partial_t f = Lf + \Gamma(f, f)$$

- The non-linearity is orthogonal to $N(L)$ :

$$\varphi \in N(L) \Longrightarrow \langle \Gamma(f, g), \varphi \rangle_{L^2_v} = 0$$

- The non-linearity can be controlled by the energy and the dissipation :

$$\langle \Gamma(f, g), h \rangle_{L^2_v} \lesssim \|h\|_{H^{s,*}_v} \left( \|g\|_{L^2_v} \|f\|_{H^{s,*}_v} + \|f\|_{L^2_v} \|g\|_{H^{s,*}_v} \right)$$

# Space homog. equation with gaussian weights

## Cauchy theory

$$\partial_t f = Lf + \Gamma(f, f)$$

- Propagation of $f_{\text{in}} \perp N(L)$:

$$f_{\text{in}} \perp N(L) \Longrightarrow f(t) \perp N(L)$$

- Good energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{L^2_v}^2 + \lambda \|f\|_{H^{s,*}}^2 \lesssim \|f\|_{L^2_v} \|f\|_{H^{s,*}}^2$$

**Conclusion**

For $f_{\text{in}} \in N(L)^\perp$ small, there exists a unique global weak solution $f(t)$ s.t.

$$\sup_{t \geq 0} \|f(t)\|_{L^2_v}^2 + \frac{\lambda}{2} \int_0^\infty \|f(t)\|_{H^{s,*}_v}^2 \mathrm{d}t \lesssim \|f_{\text{in}}\|_{L^2_v}^2$$

# Space homog. equation with gaussian weights

## The dissipated quantity

For pairwise interactions of order (distance)$^{-p}$, with $p > 2$

- Landau ($p = 2$) with $A(z) := \frac{1}{|z|}\left(I - \frac{z}{|z|} \otimes \frac{z}{|z|}\right)$

$$Q(F,F)(v) = \nabla_v \cdot \int_{\mathbb{R}^3_{v_*}} A(v-v_*)\Big[F(v_*)\nabla_v F(v) - \nabla_v F(v_*)F(v)\Big]\mathrm{d}v_*$$

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- Boltzmann ($p > 2$): differential-like $(|v - v'| \approx \theta |v - v_*|, b \approx \theta^{-2-2s})$

$$Q(F,F)(v) \approx \int_{\mathbb{R}^3_{v_*}} \mathrm{d}v_* |v - v_*|^{\gamma+2s+2} \int_{\mathbb{S}^2_\sigma} \mathrm{d}\sigma \frac{F(v'_*)F(v') - F(v_*)F(v)}{|v - v'|^{2+2s}}$$

analog to

$$(-\Delta_v)^s F(v) \approx \int_{\mathbb{R}^3_{v'}} \frac{F(v') - F(v)}{|v - v'|^{3+2s}} \mathrm{d}v'$$

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## The dissipated quantity

Optimal comparison with classical fractional Sobolev spaces :

$$\|\langle v\rangle^{\frac{\gamma}{2}+s} f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} f\|^2_{H^s_v} \lesssim \|f\|^2_{H^{s,*}} \lesssim \|\langle v\rangle^{\frac{\gamma}{2}+s} f\|^2_{H^s_v}$$

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Landau case ($s = 1$), denoting $\mathbf{P}(v) = \frac{v}{|v|} \otimes \frac{v}{|v|}$:

$$\|f\|^2_{H^{1,*}_v} = \|\langle v\rangle^{\frac{\gamma}{2}+1} f\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_v f\|^2_{L^2} + \|\langle v\rangle^{\frac{\gamma}{2}+1} (I - \mathbf{P}) \nabla_v f\|^2_{L^2}$$

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Boltzmann case ($0 < s < 1$) :

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A.M.U.X.Y. ($|v-v'| \approx \theta |v-v_*|$ and $b \approx \theta^{-2-2s}$)

$$\|f\|^2_{H^{s,*}} = \|\langle v\rangle^{\frac{\gamma}{2}+s} f\|^2_{L^2_v} + \int_{v,v_*,\sigma} |v-v_*|^\gamma b(\cos\theta)(f(v)-f(v'))^2 \mathrm{d}v_* \mathrm{d}v \mathrm{d}\sigma$$

$$\approx \|\langle v\rangle^{\frac{\gamma}{2}+s} f\|_{L^2_v}{}^2 + \int_{v,v_*} \mathrm{d}v_* \mathrm{d}v |v-v_*|^{\gamma+2s+1} \int_\sigma \mathrm{d}\sigma \frac{(f(v)-f(v'))^2}{|v-v'|^{2+2s}}$$

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$\oplus$ incorporates collision kernel $\rightarrow$ easy to estimate $\Gamma$ using $L^2$ and $H^{s,*}_v$

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Gressman-Strain, denoting the distance $\delta(v,v') = \left|\left(v, \frac{|v|^2}{2}\right) - \left(v', \frac{|v'|^2}{2}\right)\right|$ :

$$\|f\|^2_{H^{s,*}} = \|\langle v\rangle^{\frac{\gamma}{2}+s} f\|^2_{L^2_v} + \int_{\delta(v,v')\le 1} \sqrt{\langle v\rangle\langle v'\rangle}^{\gamma+2s+1} \frac{(f(v)-f(v'))^2}{\delta(v,v')^{2+2s}} \mathrm{d}v'\mathrm{d}v$$

$$\approx \|\langle v\rangle^{\frac{\gamma}{2}+s} f\|_{L^2_v} + \|\langle v\rangle^{\gamma/2+s}(I-\Delta_P)^{s/2} f\|_{L^2}$$

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$\oplus$ explicit and uses collisional geometry

$\ominus$ studied using Littlewood-Paley-type decomposition on the paraboloid $\{(v, |v|^2/2) \ : \ v \in \mathbb{R}^3\}$

# Space inhom. equation with gaussian weights

## The missing macroscopic quantities

$$\partial_t f = (L - v \cdot \nabla_x) f + \Gamma(f, f)$$

Difference with spatially homogeneous case :

$$f_{\text{in}} \perp N(L) \not\Longrightarrow f(t) \perp N(L)$$

**Problem:** no control of $\pi f$ = projection on $N(L)$ :

$$\Big\langle (L - v \cdot \nabla_x) f, f \Big\rangle_{H^2_x L^2_v} \lesssim -\|f - \pi f\|^2_{H^2_x L^2_v}$$

But $v \cdot \nabla_x$ is not antisymmetric for every inner product $\rightarrow$ find a suitable one

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## The missing macroscopic quantities

**Hypocoercivity:** find $(\cdot,\cdot)_{H^1_x L^2_v} = \langle \cdot,\cdot \rangle_{H^1_x L^2_v} + \varepsilon \langle *, \star \rangle_{H^1_x L^2_v}$ such that

$$\Big((L - v \cdot \nabla_x) f, f\Big)_{H^1_x L^2_v} \lesssim -\varepsilon \|\pi f\|^2_{H^1_x L^2_v} - \|f - \pi f\|^2_{H^1_x H^{s,*}_v}$$

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Impossible because of the dispersion coming from $v \cdot \nabla_x$

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Impossible because of the dispersion coming from $v\cdot\nabla_x$

**However** we can find $(\cdot,\cdot)_{H^1_x L^2_v} = \langle \cdot,\cdot\rangle_{H^1_x L^2_v} + \varepsilon\langle *,\star\rangle_{H^1_x L^2_v}$ such that

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$$=: -\|f\|^2_{\mathcal{H}}$$

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**Difficulty:** $L - v\cdot\nabla_x$ does not control $\|\pi f\|_{L^2_x L^2_v} \to$ avoid it:

$$(\Gamma(f,g),h)_{H^2_x L^2_v} \lesssim \|h\|_{\mathcal{H}}(\|f\|_{H^2_x L^2_v}\|g\|_{\mathcal{H}} + \|g\|_{H^2_x L^2_v}\|f\|_{\mathcal{H}})$$

# From gaussian weights to polynomial weights

**Problem:** Linearization $F = M + M^{1/2} f$ too strong/not physical
**Better:** $F$ with finite mass/energy, i.e.

$$\int F(1 + |v|^2) \mathrm{d}v < \infty$$

New linearization $F = M + \langle v \rangle^{-k} f$

$$\partial_t f = \mathbf{L} f - v \cdot \nabla_x f + \mathbf{\Gamma}(f, f)$$

$$\mathbf{L} f = \langle v \rangle^k \left\{ Q(M, \langle f \rangle^{-k} f) + Q(\langle f \rangle^{-k} f, M) \right\}$$

$$\mathbf{\Gamma}(f, g) = \langle v \rangle^k Q(\langle v \rangle^{-k} f, \langle f \rangle^{-k} g)$$

# From gaussian weights to polynomial weights

- $\mathbf{L}$ is not self-adjoint **but** satisfies for a norm $\mathbf{H}_v^{s,*}$ similar to $H_v^{s,*}$

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- Write $F = M + M^{1/2} g + \langle v\rangle^{-k} h$ and consider the system

$$\begin{cases} \partial_t h = \mathbf{B} h + \mathbf{\Gamma}(h,h) + \text{coupling terms}, \quad h_{\text{in}} = f_{\text{in}}, \\ \partial_t g = (L - v\cdot\nabla_x) g + \Gamma(g,g) + \chi h, \quad g_{\text{in}} = 0, \end{cases}$$

where $\chi = \chi_0 M^{-1/2}\langle v\rangle^k$

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**Theorem (Carrapatoso, G. — 2022)**

*For any $f_{in} = F_{in} - M$ small in $H^3_x L^2_v\Big(\langle v \rangle^{2k} \mathrm{d}v\Big)$, with $k \gg 1$, there exists a unique global weak solution $F(t) = M + \langle v \rangle^{-k} f(t)$ such that*

$$\sup_{t \geq 0} \|f(t)\|^2_{H^3_x L^2_v} + \int_0^\infty \Big\{ \|\nabla_x \pi f(t)\|^2_{H^2_x L^2_v} + \|\pi^\perp f(t)\|^2_{H^3_x \mathbf{H}^{s,*}_v} \Big\} \mathrm{d}t \lesssim \|f_{in}\|^2_{H^3_x L^2_v}$$

**Thank you for your attention**