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Hydrodynamic modes and limits of the Boltzmann equation in large functional spaces

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ENS | PSL

Introduction

The Boltzmann equation

- Rareified gas, consider only interaction between pairs of particles
- $F(t, x, v)$ = density of particles at $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$

$$\underbrace{\partial_t F + v \cdot \nabla_x F}_{\text{straight path in the } v\text{-direction}} = \underbrace{Q\Big(F(t,x,\cdot), F(t,x,\cdot)\Big)}_{\text{pairwise interactions}}(v)$$

- Homogeneous equilibria:

$$M_{R,U,T}(v) = \frac{R}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-U|^2}{2T}\right)$$

Introduction

The Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q\Big(F(t,x,\cdot), F(t,x,\cdot)\Big)(v)$$

- Macroscopic observables:
  - Mass: $R(t,x) = \int F \mathrm{d}v$
  - Momentum: $RU(t,x) = \int vF \mathrm{d}v$
  - Energy: $R|U|^2 + 3RT = \int |v|^2 F \mathrm{d}v$
- Micro conservation $\Rightarrow Q \perp 1, v, |v|^2 \Rightarrow$ Macro conservation.

Introduction

Incompressible Navier–Stokes limit

1. Many collisions in a fluid:

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon} Q(F,F)$$

2. Meso. scale (Boltzmann) $\ll$ Macro. scale (Navier-Stokes)

$$\varepsilon \partial_t F + v \cdot \nabla_x F = \frac{1}{\varepsilon} Q(F,F)$$

3. Small fluctuation around rest state $M = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}}$: $F =: M + \varepsilon f$

$$\partial_t f = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f + \frac{1}{\varepsilon} Q(f,f),$$
$$\mathcal{L} h := \frac{1}{2}\Big(Q(M,h) + Q(h,M)\Big)$$

$$\ker(\mathcal{L}) = \left\{\rho + u \cdot v + \tfrac{1}{2}(|v|^2 - 3)\theta \mid \rho, \theta \in \mathbb{R},\ u \in \mathbb{R}^3\right\} M(v)$$

Introduction

Incompressible Navier–Stokes limit

Theorem

$f^\varepsilon(0,x,v) = f_{in}(x,v)$ has finite mass/energy/entropy $\Rightarrow$ there is at least one weak global solution to

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v\cdot\nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$$

and $f^\varepsilon \rightharpoonup f$, where

$$f(t,x,v) = \left(\rho(t,x) + u(t,x)\cdot v + \frac{1}{2}(|v|^2 - 3)\theta(t,x)\right) M(v)$$

with ρ, u, θ solving

$$\begin{aligned} \partial_t u + u\cdot\nabla u = \mu\Delta u, &\quad \partial_t\theta + u\cdot\theta = \kappa\Delta\theta, \\ \nabla\cdot u = 0, &\quad \nabla(\rho+\theta) = 0, \end{aligned}$$

Introduction

Partial timeline

Q: Strong solution/uniqueness/limit ? $\to$ Functional space ?

- [Grad65] study $\mathcal{L}$ in $L^2_v\left(M^{-1}(v)\mathrm{d}v\right)$
- [Ellis-Pinsky75] spectrum of $\mathcal{L} + v \cdot \nabla_x$ for low frequencies in $L^2_v H^s_x \left(M^{-1}(v)\mathrm{d}v\right) \Rightarrow$ asymptotic behavior of $\frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right)$
- [Ukai86]
  - spectrum for large frequencies
  - perturbative strong solutions for Boltzmann
- [Bardos-Ukai91] convergence of Ukai's solutions

Q: Only assume $\int f(1+|v|^2)\mathrm{d}v < \infty$ (finite mass/energy) ?

- [Gualdani-Mischler-Mouhot17]
  - study of $\mathcal{L}$ in $L^2_v\left((1+|v|^k)\mathrm{d}v\right)$
  - perturbative solutions in $L^2_v H^s_x\left((1+|v|^k)\mathrm{d}v\right)$
- [Briant-Merino-Mouhot19] weak hydrodynamic limit

Introduction

General stategy

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v\cdot\nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),$$

$$\Big\downarrow \text{Duhamel with } f^\varepsilon_{|t=0} = f_{\text{in}} \quad U^\varepsilon(t) := \exp\Big(\tfrac{1}{\varepsilon^2}(\mathcal{L}+\varepsilon v\cdot\nabla_x)\Big)$$

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-t') Q(f^\varepsilon(t'), f^\varepsilon(t'))\mathrm{d}t',$$

$$\Big\downarrow \text{solve + let } \varepsilon \to 0$$

$$f(t) = U(t) f_{\text{in}} + \Psi(t)(f, f),$$

This requires

- Bounds on U^ε+Duhamel term
- Asymptotics of U^ε+Duhamel term

$\Rightarrow$ study spectrum of $\varepsilon^{-2}\,(\mathcal{L} + \varepsilon v\cdot\nabla_x)$

$\Rightarrow$ study spectrum of $\mathcal{L}(\xi) := \mathcal{L} + iv\cdot\xi$ for $\xi \approx 0$

Spectral study

Properties of the homogeneous linearized operator

(H1) $\|vf\| \lesssim \|\mathcal{L}f\| + \|f\|$

(H2) $\Sigma(\mathcal{L}) \cap \{\Re\mathrm{e} z > -a\} = \{0\}$ for some $a > 0$

(H3) 0 is a fivefold $\frac{1}{2}$-simple eigenvalue

Theorem

In $L^2\left(M^{-1}\mathrm{d}v\right)$ and $L^2\left(\langle v\rangle^k\mathrm{d}v\right)$, (H1)-(H3) are satified

Spectral study

Properties of the homogeneous linearized operator

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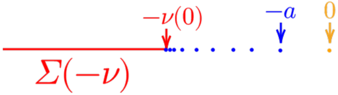
Proof in $L^2\left(M^{-1}\mathrm{d}v\right)$*:* $\mathcal{L}$ self-adjoint + $\mathcal{L} = -\nu(v) + K$

- ν continuous + $\nu(v) \approx 1 + |v|$
- K compact

Self-adjointness + Weyl's theorem $\Rightarrow$

$\Sigma(\mathcal{L}) =$ 

Self-adjointness $\Rightarrow$ 0 is $\frac{1}{2}$ semi-simple

N.B.: a made explicit by C. Baranger and C. Mouhot in 2005

Spectral study

Properties of the homogeneous linearized operator

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(H2) $\Sigma(\mathcal{L}) \cap \{\Re \mathrm{e} z > -a\} = \{0\}$ for some $a > 0$

(H3) 0 is a fivefold $\frac{1}{2}$-simple eigenvalue

Proof in $L^2\left(\langle v\rangle^k \mathrm{d}v\right)$*:* $\mathcal{L}$ **not self-adjoint**, $\mathcal{L} = \mathcal{B} + \mathcal{A}$

- $\mathcal{B} = -\nu +$ "small" operator $\Rightarrow$

$$\Sigma(\mathcal{B}) \subset \{\Re e z \leq -a := -\nu(0) + \delta\}$$

- $\mathcal{A} : L^2\left(\langle v\rangle^k \mathrm{d}v\right) \xrightarrow{bded} L^2\left(M^{-1}\mathrm{d}v\right)$

Deduce from the $L^2\left(M^{-1}\mathrm{d}v\right)$ case by factorization:

$$(z-\mathcal{L})^{-1} f = (z-\mathcal{B})^{-1} f + (z-\mathcal{L})^{-1} \mathcal{A}(z-\mathcal{B})^{-1} f$$

Spectral study

Spectrum of the inhomogeneous linearized operator

Theorem

For $0 < \delta \ll 1$ *and* $|\xi| \ll 1$

1. $\Sigma(\mathcal{L}(\xi)) \cap \{\Re e z > -a + \delta\} = \{\lambda^{\pm}_{disp}(\xi), \lambda_{Bou}(\xi), \lambda_{inc}(\xi)\}$
2. λ_{inc} *is twofold,* λ_{Bou} *and* $\lambda^{\pm}_{disp}$ *are simple*
3. *The following expansions hold*

$$\lambda^{\pm}_{disp}(\xi) \approx \pm ic|\xi| - \kappa|\xi|^2$$

$$\lambda_{Bou}(\xi) \approx -\kappa|\xi|^2, \quad \lambda_{inc}(\xi) \approx -\mu|\xi|^2$$

$$\partial_t u + u \cdot \nabla u = \mu \Delta u, \quad \partial_t \theta + u \cdot \theta = \kappa \Delta \theta,$$
$$\nabla \cdot u = 0, \qquad \nabla(\rho + \theta) = 0,$$

$$\mathcal{L}(\xi) = \mathcal{L} + iv \cdot \xi,$$

Spectral study

Spectrum of the inhomogeneous linearized operator

Theorem

The corresponding projectors satisfy

$$\mathcal{P}_*(\xi) \approx \mathcal{P}_*^{(0)}\left(\frac{\xi}{|\xi|}\right) + \xi \cdot \mathcal{P}_*^{(1)}\left(\frac{\xi}{|\xi|}\right)$$

where $\mathcal{P}_*^0$ are proj. on subspaces of $\{\rho + u \cdot v + \frac{1}{2}(|v|^2 - 3)\theta\}M(v)$

$$\mathcal{P}_{inc}^{(0)}\left(\frac{\xi}{|\xi|}\right) = \textit{Proj. on } \{u \perp \xi,\ \rho = \theta = 0\},$$

$$\mathcal{P}_{Bou}^{(0)}\left(\frac{\xi}{|\xi|}\right) = \textit{Proj. on } \{u = 0,\ \rho + \theta = 0\},$$

$$\mathcal{P}_{disp}^{0,\pm}\left(\frac{\xi}{|\xi|}\right) = \textit{Proj. on } \left\{u = \pm\frac{\xi}{|\xi|},\ \theta = \frac{2}{3}\rho\right\}$$

$$\partial_t u + u \cdot \nabla u = \mu \Delta u, \qquad \partial_t \theta + u \cdot \theta = \kappa \Delta \theta,$$

$$\nabla \cdot u = 0, \ \text{ i.e. } \ \widehat{u}(\xi) \perp \xi, \qquad \nabla(\rho + \theta) = 0,$$

Spectral study

Spectrum of the inhomogeneous linearized operator

Idea of proof for strong hydrodynamic limit:

$$\partial_t u + u \cdot \nabla u = \mu \Delta u$$

$$\downarrow$$

$$u(t) = e^{t\mu\Delta} \mathbb{P} u_{\text{in}} - \int_0^t e^{(t-t')\mu\Delta} \nabla \cdot \mathbb{P}\left(u \otimes u\right)(t') \mathrm{d}t'$$

Plug expansions in integral formulation of Boltzmann:

$$\begin{aligned} f^\varepsilon(t) =& U^\varepsilon(t) f_{\text{in}} + \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-t') Q(f^\varepsilon, f^\varepsilon)(t') \mathrm{d}t' \\ \approx& e^{t\mu\Delta} \mathcal{P}^{(0)}_{\text{inc}} f_{\text{in}} + \frac{1}{\varepsilon}\int_0^t e^{(t-t')\mu\Delta} \left(\mathcal{P}^{(0)}_{\text{inc}} + \varepsilon \nabla \cdot \mathcal{P}^{(1)}_{\text{inc}}\right) Q(f^\varepsilon, f^\varepsilon)(t') \mathrm{d}t' \\ &+ \dots \\ =& e^{t\mu\Delta} (\mathbb{P} u_{\text{in}}) \cdot M(v) + \int_0^t e^{(t-t')\mu\Delta} \nabla \cdot \mathcal{P}^{(1)}_{\text{inc}} Q(f^\varepsilon, f^\varepsilon)(t') \mathrm{d}t' + \dots \end{aligned}$$

Spectral study

Spectrum of the inhomogeneous linearized operator

Strategy:

$$\mathcal{L}(\xi) = \sum_{*} \lambda_*(\xi)\mathcal{P}(\xi) + \mathcal{L}^{\perp}(\xi), \tag{1}$$

$$\text{Space} = \bigoplus_{*} X_* \oplus Y, \tag{2}$$

$$\mathcal{P}_*(\xi) = \text{proj of } X_*, \quad \mathcal{L}^{\perp}(\xi) = \text{op. on } Y \tag{3}$$

$$\Sigma(\mathcal{L}^{\perp}(\xi)) \subset \{\Re e z \leq -a + \delta\} \tag{4}$$

$$+ \text{ expansions of } \lambda_*(\xi) \text{ and } \mathcal{P}_*(\xi) \tag{5}$$

Steps of the proof

1. Localization of the spectrum: the eigenvalue 0 of $\mathcal{L}$ splits as $2 + 1 + 1$ eigenvalues of $\mathcal{L}(\xi) \Rightarrow$ (1)–(4)
2. Kato's reduction process $\Rightarrow$ (5)+distinguish eigenvalues

Spectral study

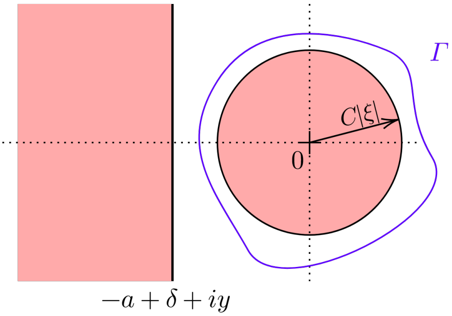
Spectrum of the inhomogeneous linearized operator

Step 1: Localization of the spectrum.

Assume $0 < |\xi| \leq A|z|$ and $\Re e z > -a + \delta$

$$(z - \mathcal{L}(\xi))^{-1} = \underbrace{(z - \mathcal{L})^{-1}}_{\mathcal{O}(1/|z|)} \Big(- i\xi \cdot \underbrace{v\,(z - \mathcal{L})^{-1}}_{\mathcal{O}(1+1/|z|)} + \mathrm{Id}\Big)^{-1}$$

$$= \mathcal{O}\left(\frac{1}{|\xi|}\right) \times (1 - \mathcal{O}(A))^{-1}$$



Spectral study

Spectrum of the inhomogeneous linearized operator

Step 2: Nature of this spectrum.
Consider Γ encircling $\{|z| \leq C|\xi|\}$ for $|\xi| \ll 1$
Consider projection associated with this portion of the spectrum:

$$\begin{aligned}\mathcal{P}(\xi) &:= \frac{1}{2i\pi}\int_\Gamma (z-\mathcal{L}(\xi))^{-1}\,\mathrm{d}z \\ &= \frac{1}{2i\pi}\int_\Gamma (z-\mathcal{L})^{-1}\,\mathrm{d}z + \mathcal{O}(\xi) =: \mathcal{P} + \mathcal{O}(\xi)\end{aligned}$$

$\mathcal{P}$ =projection on $\ker(\mathcal{L}) = \{1, v_1, v_2, v_3, \left(|v|^2-3\right)/2\}$.

Lemma

Let P_1, P_2 be projectors, $\|P_1 - P_2\| < 1 \Rightarrow \mathrm{R}(P_1) \cong \mathrm{R}(P_2)$ through

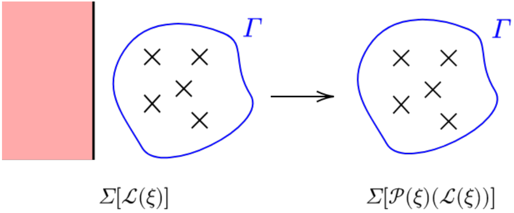
$$U = \left(P_1P_2 + P_1^\perp P_2^\perp\right)\left(1-(P_1-P_2)^2\right)^{-1/2}$$

$\Rightarrow \mathcal{P}(\xi)$ has rank five $\Rightarrow \Gamma$ encircles 5 eigenvalues (with multiplicity)

Spectral study

Spectrum of the inhomogeneous linearized operator

Step 3: Expansion of eigenvalues/functions



$$\mathcal{P}(\xi)\mathcal{L}(\xi) = \text{op. of } \mathrm{R}\left(\mathcal{P}(\xi)\right), \text{ spectrum} = \Sigma(\mathcal{L}(\xi)) \cap \{\Re e z > -a + \delta\}$$

$$\Downarrow \mathcal{U}(\xi)\Big(\ker(\mathcal{L})\Big) = \mathrm{R}(\mathcal{P}(\xi))$$

$$\mathcal{U}(\xi)^{-1}\mathcal{P}(\xi)\mathcal{L}(\xi)\mathcal{U}(\xi) = \text{op. of} \ker(\mathcal{L}), \text{ same spectrum}$$

Spectral study

Spectrum of the inhomogeneous linearized operator

- Eigenvalues = $\mathcal{O}(|\xi|)$ $\Rightarrow$ consider instead

$$L(\xi) := \frac{1}{|\xi|}\mathcal{U}(\xi)^{-1}\mathcal{P}(\xi)\mathcal{L}(\xi)\mathcal{U}(\xi)$$

N.B.: Converges for $\xi \to 0 \Leftrightarrow 0 = \frac{1}{2}$-simple eigenvalue of $\mathcal{L}$

- Symetries of $\mathcal{L}$ $\Rightarrow$ stable subspaces

$$L(\xi) = \begin{pmatrix} \frac{1}{|\xi|}\lambda_{\text{inc}}(\xi)\mathrm{Id} & 0 \\ 0 & L'(\xi) \end{pmatrix},$$

$$\{u \perp \xi,\ \rho = \theta = 0\} \oplus \{\rho + \theta = 0,\ u = 0\} \oplus \left\{\theta = \frac{2}{3}\rho,\ u = \pm\frac{\xi}{|\xi|}\rho\right\}$$

- $L'(\xi) \xrightarrow{\xi \to 0} \mathrm{diag}(ic, 0, -ic)$ $\Rightarrow$ usual perturbation theory for matrices $\Rightarrow$ expansion of eigenvalues/eigenfunctions

Spectral study

Spectrum of the inhomogeneous linearized operator

$$\mathcal{L}(\xi) = \mathcal{P}(\xi)\mathcal{L}(\xi) + \mathcal{P}^{\perp}(\xi)\mathcal{L}(\xi)$$

$$\Downarrow \begin{array}{l} L(\xi) = \sum_* \frac{1}{|\xi|}\lambda_*(\xi) \times P_*(\xi) \\ L(\xi) = \frac{1}{|\xi|}\mathcal{U}(\xi)^{-1}\mathcal{P}(\xi)\mathcal{L}(\xi)\mathcal{U}(\xi) \end{array}$$

$$\mathcal{L}(\xi) = \sum_* \lambda_*(\xi)\mathcal{P}_*(\xi) + \mathcal{L}^{\perp}(\xi)$$

Argument valid in $L^2\left(\langle v\rangle^k \mathrm{d}v\right)$ and $L^2\left(M^{-1}\mathrm{d}v\right) \Rightarrow$

$$\mathcal{P}_*(\xi) : L^2(\langle v\rangle^k \mathrm{d}v) \xrightarrow{\mathcal{U}(\xi)^{-1}\mathcal{P}(\xi)} \overset{\circlearrowleft P_*(\xi)}{\ker(\mathcal{L})} \xrightarrow{\mathcal{U}(\xi)} L^2\left(M^{-1}\mathrm{d}v\right)$$

Conclusion

Summary

Theorem

For $0 < \delta \ll 1$ *and* $|\xi| \ll 1$

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$$\mathcal{P}_*(\xi) \approx \mathcal{P}^{(0)}_* \left(\frac{\xi}{|\xi|}\right) + \xi \cdot \mathcal{P}^{(1)}_* \left(\frac{\xi}{|\xi|}\right)$$

1. Localization of the spectrum: the eigenvalue 0 of $\mathcal{L}$ splits as several eigenvalues of $\mathcal{L}(\xi)$
2. Kato's reduction process

Conclusion

Comparison with the original study

Ellis & Pinsky's proof:

$\ominus$ complex, non-constructive, only Gaussian weights

$\oplus$ very general interactions between particles

Our approach:

$\ominus$ only hard-spheres

$\oplus$ polynomial weights, easier

Different scaling $\rightarrow$ different models (Euler, wave equation)

Conclusion

References

Boltzmann equation

- Ukai, Yang, *Mathematical Theory of Boltzmann Equation*, ('08).
- Baranger, Mouhot, *Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials*, ('05).

Spectral studies of the Boltzmann equation close to equilibrium

- Ellis, Pinsky, *The first and second fluid approximations of the linearized Boltzmann equation*, ('75).
- Gervais, *A spectral study of the linearized Boltzmann operator in L^2-spaces with polynomial and Gaussian weights*, ('21).

Hydrodynamic limits

- Bardos, Golse, Levermore, *Fluid dynamic limits of kinetic eq. I*, ('91).
- Bardos, Ukai, *The classical incompressible Navier-Stokes limit of the Boltzmann equation*, ('91).

Enlargement theory

- Gualdani, Mischler, Mouhot, *Factorization for non-symmetric op. and exponential H-theo.*, ('17).
- Briant, Merino-Aceituno, Mouhot, *From Boltzmann to incompressible Navier- Stokes in Sobolev spaces with polynomial weight*, ('19).

Spectral/perturbation theory

- Kato, *Perturbation Theory for Linear Operators*, ('66).

Thanks for your attention!