# Markov Decision Processes and Dynamic Programming

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# In This Lecture

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- **How do we formalize the agent-environment interaction?**

$\Rightarrow$ *Markov Decision Process (MDP)*

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- **How do we formalize the agent-environment interaction?**

  $\Rightarrow$ *Markov Decision Process (MDP)*

- **How do we solve an MDP?**

  $\Rightarrow$ *Dynamic Programming*

# Outline

# Probability Theory

## Definition (Conditional probability)

*Given two events $A$ and $B$ with $\mathbb{P}(B) > 0$, the* ***conditional probability*** *of $A$ given $B$ is*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cup B)}{\mathbb{P}(B)}.$$

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*Similarly, if $X$ and $Y$ are non-degenerate and jointly continuous random variables with density $f_{X,Y}(x,y)$ then if $B$ has positive measure then the **conditional probability** is*

$$\mathbb{P}(X \in A | Y \in B) = \frac{\int_{y \in B} \int_{x \in A} f_{X,Y}(x,y) dxdy}{\int_{y \in B} \int_{x} f_{X,Y}(x,y) dxdy}.$$

# Probability Theory

**Definition (Law of total expectation)**

*Given a function f and two random variables $X, Y$ we have that*

$$\mathbb{E}_{X,Y}\big[f(X,Y)\big] = \mathbb{E}_X\Big[\mathbb{E}_Y\big[f(x,Y)|X=x\big]\Big].$$

# Norms and Contractions

**Definition**

*Given a vector space $\mathcal{V} \subseteq \mathbb{R}^d$ a function $f : \mathcal{V} \to \mathbb{R}_0^+$ is a **norm** if an only if*

- *If $f(v) = 0$ for some $v \in \mathcal{V}$, then $v = 0$.*
- *For any $\lambda \in \mathbb{R}, v \in \mathcal{V}$, $f(\lambda v) = |\lambda| f(v)$.*
- *Triangle inequality: For any $v, u \in \mathcal{V}$, $f(v + u) \leq f(v) + f(u)$.*

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- $L_{2,P}$-matrix norm ($P$ is a positive definite matrix)

$$||v||_P^2 = v^\top P v.$$

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## Definition

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## Definition

*A vector space $\mathcal{V}$ equipped with a norm $||\cdot||$ is complete if every Cauchy sequence in $\mathcal{V}$ is convergent in the norm of the space.*

# Norms and Contractions

## Definition

*An operator* $\mathcal{T} : \mathcal{V} \rightarrow \mathcal{V}$ *is L-Lipschitz if for any* $v, u \in \mathcal{V}$

$$||\mathcal{T}v - \mathcal{T}u|| \leq L||u - v||.$$

*If* $L \leq 1$ *then* $\mathcal{T}$ *is a non-expansion, while if* $L < 1$ *then* $\mathcal{T}$ *is a L-contraction.*
*If* $\mathcal{T}$ *is Lipschitz then it is also continuous, that is*

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**Definition**

*A vector* $v \in \mathcal{V}$ *is a fixed point of the operator* $\mathcal{T} : \mathcal{V} \rightarrow \mathcal{V}$ *if* $\mathcal{T}v = v$.

# Norms and Contractions

**Proposition (Banach Fixed Point Theorem)**

Let $\mathcal{V}$ be a *complete* vector space equipped with the norm $||\cdot||$ and $\mathcal{T}:\mathcal{V}\rightarrow\mathcal{V}$ be a *$\gamma$-contraction* mapping. Then

1. $\mathcal{T}$ admits a *unique fixed point* $v$.
2. For any $v_0 \in \mathcal{V}$, if $v_{n+1} = \mathcal{T} v_n$ then $v_n \rightarrow_{||\cdot||} v$ with a *geometric convergence rate*:

$$||v_n - v|| \leq \gamma^n ||v_0 - v||.$$

# Linear Algebra

Given a square matrix $A \in \mathbb{R}^{N \times N}$:

- *Eigenvalues of a matrix (1).* $v \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$ are *eigenvector* and *eigenvalue* of $A$ if

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- *Matrix inversion.* $A$ can be *inverted* if and only if $\forall i,\ \lambda_i \neq 0$.

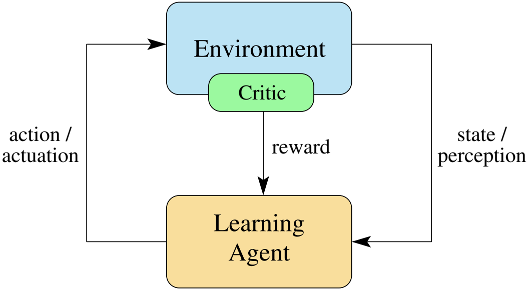
# Linear Algebra

- *Stochastic matrix.* A square matrix $P \in \mathbb{R}^{N \times N}$ is a stochastic matrix if
  1. all non-zero entries, $\forall i, j$, $[P]_{i,j} \geq 0$
  2. all the rows sum to one, $\forall i$, $\sum_{j=1}^{N} [P]_{i,j} = 1$.

  All the eigenvalues of a stochastic matrix are bounded by 1, i.e., $\forall i$, $\lambda_i \leq 1$.

# Outline

# The Reinforcement Learning Model



# Markov Chains

**Definition (Markov chain)**

*Let the state space $X$ be a bounded compact subset of the Euclidean space, the discrete-time dynamic system $(x_t)_{t \in \mathbb{N}} \in X$ is a Markov chain if it satisfies the Markov property*

$$\mathbb{P}(x_{t+1} = x \,|\, x_t, x_{t-1}, \ldots, x_0) = \mathbb{P}(x_{t+1} = x \,|\, x_t),$$

*Given an initial state $x_0 \in X$, a Markov chain is defined by the transition probability $p$*

$$p(y|x) = \mathbb{P}(x_{t+1} = y | x_t = x).$$

# Example: Weather prediction

*Informal definition*: we want to describe how the weather evolves over time.
$\Rightarrow$ Board!

# Markov Decision Process

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$$p(y|x, a) = \mathbb{P}(x_{t+1} = y | x_t = x, a_t = a),$$

- $r(x, a, y)$ *is the reward of transition* $(x, a, y)$.

# Examples

- Park a car
- Find the shortest path from home to school
- Schedule a fleet of truck

# Policy

**Definition (Policy)**

*A decision rule* $\pi_t$ *can be*

- *Deterministic:* $\pi_t : X \rightarrow A$,
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- *Non-stationary:* $\pi = (\pi_0, \pi_1, \pi_2, \dots)$,
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*Remark*: MDP $M$ + stationary policy $\pi \Rightarrow$ *Markov chain* of state $X$ and transition probability $p(y|x) = p(y|x, \pi(x))$.

# Question

*Is the MDP formalism powerful enough?*

$\Rightarrow$ *Let's try!*

# Example: the Retail Store Management Problem

*Description.* At each month $t$, a store contains $x_t$ *items* of a specific goods and the demand for that goods is $D_t$. At the end of each month the manager of the store can *order* $a_t$ more items from his supplier. Furthermore we know that

- The *cost* of maintaining an inventory of $x$ is $h(x)$.
- The *cost* to order $a$ items is $C(a)$.
- The *income* for selling $q$ items is $f(q)$.
- If the demand $D$ is bigger than the available inventory $x$, customers that cannot be served leave.
- The *value of the remaining inventory* at the end of the year is $g(x)$.
- *Constraint*: the store has a maximum capacity $M$.

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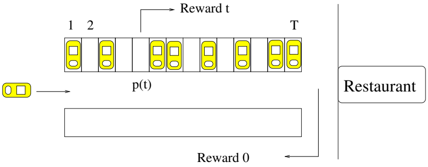
- The demand $D_t$ is *stochastic and time-independent*. Formally, $D_t \overset{i.i.d.}{\sim} \mathcal{D}$.
- *Reward*: $r_t = -C(a_t) - h(x_t + a_t) + f([x_t + a_t - x_{t+1}]^+)$.

# Exercise: the Parking Problem

A driver wants to park his car as close as possible to the restaurant.



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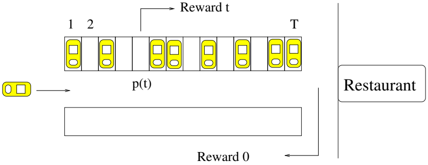


- The driver cannot see whether a place is available unless he is in front of it.
- There are $P$ places.
- At each place $i$ the driver can either move to the next place or park (if the place is available).
- The closer to the restaurant the parking, the higher the satisfaction.
- If the driver doesn't park anywhere, then he/she leaves the restaurant and has to find another one.

# Question

*How do we evaluate a policy and compare two policies?*

$\Rightarrow$ *Value function!*

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- *Infinite time horizon with average reward*: the problem never terminates but the agent only focuses on the (expected) *average of the rewards*.

# State Value Function

- *Finite time horizon* $T$: deadline at time $T$, the agent focuses on the sum of the rewards up to $T$.

$$V^\pi(t,x) = \mathbb{E}\Big[\sum_{s=t}^{T-1} r(x_s, \pi_s(x_s)) + R(x_T) \Big| x_t = x; \pi\Big],$$

where $R$ is a value function for the final state.

# State Value Function

- *Infinite time horizon with discount*: the problem never terminates but rewards which are *closer* in time receive a *higher* importance.

$$V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big],$$

with discount factor $0 \leq \gamma < 1$:

  - *small* = short-term rewards, *big* = long-term rewards
  - for any $\gamma \in [0, 1)$ the series always converge (for bounded rewards)

# State Value Function

- *Infinite time horizon with terminal state*: the problem never terminates but the agent will eventually reach a *termination state*.

$$V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{T} r(x_t, \pi(x_t)) | x_0 = x; \pi\Big],$$

where $T$ is the first (*random*) time when the *termination state* is achieved.

# State Value Function

- *Infinite time horizon with average reward*: the problem never terminates but the agent only focuses on the (expected) *average of the rewards*.

$$V^\pi(x) = \lim_{T\to\infty} \mathbb{E}\Big[\frac{1}{T}\sum_{t=0}^{T-1} r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big].$$

# State Value Function

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A non-stationary policy $\pi$ applied from state $x_0$ returns

$$(x_0, r_0, x_1, r_1, x_2, r_2, \ldots)$$

with $r_t = r(x_t, \pi_t(x_t))$ and $x_t \sim p(\cdot | x_{t-1}, a_t = \pi(x_t))$ are *random* realizations.

The value function (discounted infinite horizon) is

$$V^\pi(x) = \mathbb{E}_{(x_1, x_2, \ldots)}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big],$$

# Optimal Value Function

**Definition (Optimal policy and optimal value function)**

*The solution to an MDP is an optimal policy $\pi^*$ satisfying*

$$\pi^* \in \arg\max_{\pi \in \Pi} V^\pi$$

*in all the states $x \in X$, where $\Pi$ is some policy set of interest.*

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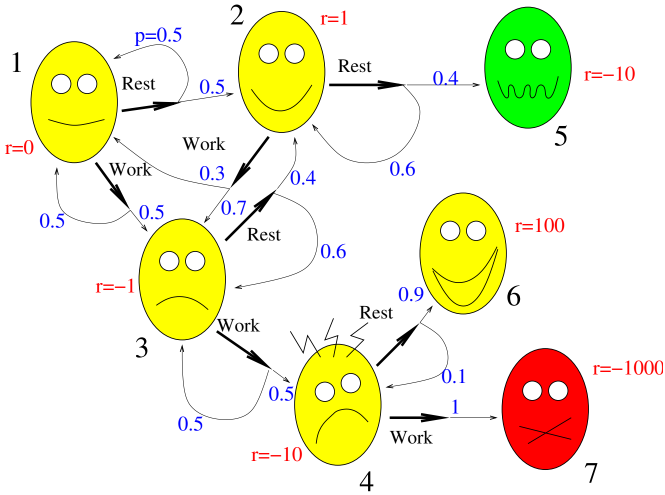
*Remark*: $\pi^* \in \arg\max(\cdot)$ and not $\pi^* = \arg\max(\cdot)$ because an MDP may admit more than one optimal policy.

# Example: the EC student dilemma



# Example: the EC student dilemma

- *Model*: all the transitions are Markov, states $x_5, x_6, x_7$ are terminal.
- *Setting*: infinite horizon with terminal states.
- *Objective*: find the policy that maximizes the expected sum of rewards before achieving a terminal state.

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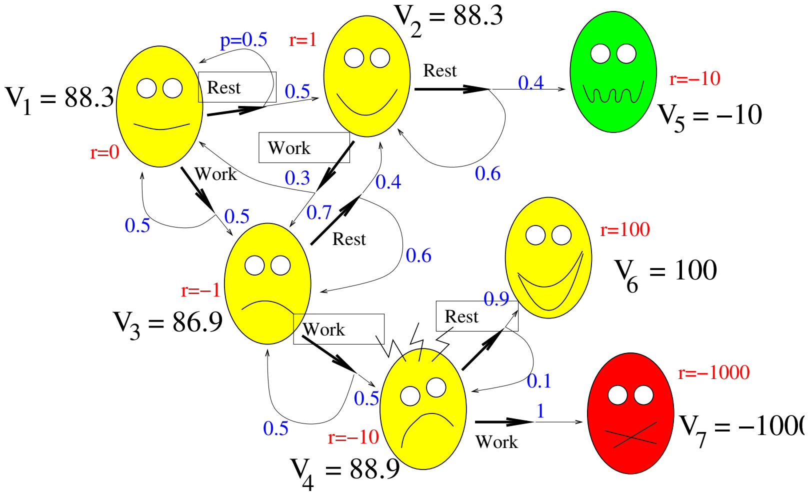

# Example: the EC student dilemma

$$
\begin{aligned}
V_7 &= -1000 \\
V_6 &= 100 \\
V_5 &= -10 \\
V_4 &= -10 + 0.9 V_6 + 0.1 V_4 \simeq 88.9 \\
V_3 &= -1 + 0.5 V_4 + 0.5 V_3 \simeq 86.9 \\
V_2 &= 1 + 0.7 V_3 + 0.3 V_1 \\
V_1 &= \max\{0.5 V_2 + 0.5 V_1, 0.5 V_3 + 0.5 V_1\} \\
V_1 &= V_2 = 88.3
\end{aligned}
$$

# State-Action Value Function

**Definition**

*In discounted infinite horizon problems, for any policy $\pi$, the* *state-action value function (or Q-function)* $Q^\pi : X \times A \mapsto \mathbb{R}$ *is*

$$Q^\pi(x, a) = \mathbb{E}\Big[\sum_{t\geq 0} \gamma^t r(x_t, a_t) | x_0 = x, a_0 = a, a_t = \pi(x_t), \forall t \geq 1\Big],$$

*and the corresponding optimal Q-function is*

$$Q^*(x, a) = \max_\pi Q^\pi(x, a).$$

# State-Action Value Function

The relationships between the V-function and the Q-function are:

$$\begin{aligned} Q^\pi(x,a) &= r(x,a) + \gamma \sum_{y \in X} p(y|x,a) V^\pi(y) \\ V^\pi(x) &= Q^\pi(x, \pi(x)) \\ Q^*(x,a) &= r(x,a) + \gamma \sum_{y \in X} p(y|x,a) V^*(y) \\ V^*(x) &= Q^*(x, \pi^*(x)) = \max_{a \in A} Q^*(x,a). \end{aligned}$$

# Outline

# Question

*Is there any more compact way to describe a value function?*

$\Rightarrow$ *Bellman equations!*

# The Bellman Equation

## Proposition

For any stationary policy $\pi = (\pi, \pi, \dots)$, the state value function at a state $x \in X$ satisfies the *Bellman equation*:

$$V^\pi(x) = r(x, \pi(x)) + \gamma \sum_y p(y|x, \pi(x)) V^\pi(y).$$

# The Bellman Equation

*Proof.*
For any policy $\pi$,

$$
\begin{aligned}
V^\pi(x) &= \mathbb{E}\Big[\sum_{t\geq 0}\gamma^t r(x_t,\pi(x_t))\,|\,x_0=x;\pi\Big] \\
&= r(x,\pi(x)) + \mathbb{E}\Big[\sum_{t\geq 1}\gamma^t r(x_t,\pi(x_t))\,|\,x_0=x;\pi\Big] \\
&= r(x,\pi(x)) \\
&\qquad + \gamma\sum_y \mathbb{P}(x_1=y\,|\,x_0=x;\pi(x_0))\mathbb{E}\Big[\sum_{t\geq 1}\gamma^{t-1} r(x_t,\pi(x_t))\,|\,x_1=y;\pi\Big] \\
&= r(x,\pi(x)) + \gamma\sum_y p(y|x,\pi(x))V^\pi(y).
\end{aligned}
$$

■

# The Optimal Bellman Equation

***Bellman's Principle of Optimality*** [1]:

> *"An optimal policy has the property that, whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."*

# The Optimal Bellman Equation

**Proposition**

The optimal value function $V^*$ (i.e., $V^* = \max_\pi V^\pi$) is the solution to the *optimal Bellman equation*:

$$V^*(x) = \max_{a \in A} \Big[ r(x,a) + \gamma \sum_y p(y|x,a) V^*(y) \Big].$$

and the optimal policy is

$$\pi^*(x) = \arg\max_{a \in A} \Big[ r(x,a) + \gamma \sum_y p(y|x,a) V^*(y) \Big].$$

# The Optimal Bellman Equation

*Proof.*

For any policy $\pi = (a, \pi')$ (possibly non-stationary),

$$
\begin{aligned}
V^*(x) &\overset{(a)}{=} \max_\pi \mathbb{E}\Big[\sum_{t\geq 0} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big] \\
&\overset{(b)}{=} \max_{(a,\pi')} \Big[r(x,a) + \gamma \sum_y p(y|x,a) V^{\pi'}(y)\Big] \\
&\overset{(c)}{=} \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) \max_{\pi'} V^{\pi'}(y)\Big] \\
&\overset{(d)}{=} \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) V^*(y)\Big].
\end{aligned}
$$

■

# The Bellman Operators

*Notation.* w.l.o.g. a discrete state space $|X| = N$ and $V^\pi \in \mathbb{R}^N$.

**Definition**

*For any $W \in \mathbb{R}^N$, the Bellman operator $\mathcal{T}^\pi : \mathbb{R}^N \to \mathbb{R}^N$ is*

$$\mathcal{T}^\pi W(x) = r(x, \pi(x)) + \gamma \sum_y p(y|x, \pi(x)) W(y),$$

*and the optimal Bellman operator (or dynamic programming operator) is*

$$\mathcal{T} W(x) = \max_{a \in A} \Big[ r(x, a) + \gamma \sum_y p(y|x, a) W(y) \Big].$$

# The Bellman Operators

## Proposition

Properties of the Bellman operators

1. *Monotonicity*: for any $W_1, W_2 \in \mathbb{R}^N$, if $W_1 \leq W_2$ component-wise, then

$$\begin{aligned} \mathcal{T}^\pi W_1 &\leq \mathcal{T}^\pi W_2, \\ \mathcal{T} W_1 &\leq \mathcal{T} W_2. \end{aligned}$$

# The Bellman Operators

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$$\begin{aligned} \mathcal{T}^\pi W_1 &\leq \mathcal{T}^\pi W_2, \\ \mathcal{T} W_1 &\leq \mathcal{T} W_2. \end{aligned}$$

2. *Offset*: for any scalar $c \in \mathbb{R}$,

$$\begin{aligned} \mathcal{T}^\pi (W + c I_N) &= \mathcal{T}^\pi W + \gamma c I_N, \\ \mathcal{T} (W + c I_N) &= \mathcal{T} W + \gamma c I_N, \end{aligned}$$

# The Bellman Operators

## Proposition

3. *Contraction in $L_\infty$-norm*: for any $W_1, W_2 \in \mathbb{R}^N$

$$
\begin{aligned}
||\mathcal{T}^\pi W_1 - \mathcal{T}^\pi W_2||_\infty &\leq \gamma ||W_1 - W_2||_\infty, \\
||\mathcal{T} W_1 - \mathcal{T} W_2||_\infty &\leq \gamma ||W_1 - W_2||_\infty.
\end{aligned}
$$

# The Bellman Operators

## Proposition

3. *Contraction in $L_\infty$-norm*: for any $W_1, W_2 \in \mathbb{R}^N$

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4. *Fixed point*: For any policy $\pi$

$V^\pi$ is the *unique fixed point* of $\mathcal{T}^\pi$,

$V^*$ is the *unique fixed point* of $\mathcal{T}$.

# The Bellman Operators

## Proposition

3. *Contraction in $L_\infty$-norm*: for any $W_1, W_2 \in \mathbb{R}^N$

$$\begin{aligned} \|\mathcal{T}^\pi W_1 - \mathcal{T}^\pi W_2\|_\infty &\le \gamma \|W_1 - W_2\|_\infty, \\ \|\mathcal{T} W_1 - \mathcal{T} W_2\|_\infty &\le \gamma \|W_1 - W_2\|_\infty. \end{aligned}$$

4. *Fixed point*: For any policy $\pi$

$V^\pi$ is the *unique fixed point* of $\mathcal{T}^\pi$,

$V^*$ is the *unique fixed point* of $\mathcal{T}$.

Furthermore for any $W \in \mathbb{R}^N$ and any stationary policy $\pi$

$$\begin{aligned} \lim_{k\to\infty} (\mathcal{T}^\pi)^k W &= V^\pi, \\ \lim_{k\to\infty} (\mathcal{T})^k W &= V^*. \end{aligned}$$

# The Bellman Equation

*Proof.*

The contraction property (3) holds since for any $x \in X$ we have

$$
\begin{aligned}
&|\mathcal{T} W_1(x) - \mathcal{T} W_2(x)| \\
&= \Big| \max_a \big[ r(x,a) + \gamma \sum_y p(y|x,a) W_1(y) \big] - \max_{a'} \big[ r(x,a') + \gamma \sum_y p(y|x,a') W_2(y) \big] \Big| \\
&\overset{(a)}{\leq} \max_a \Big| \big[ r(x,a) + \gamma \sum_y p(y|x,a) W_1(y) \big] - \big[ r(x,a) + \gamma \sum_y p(y|x,a) W_2(y) \big] \Big| \\
&= \gamma \max_a \sum_y p(y|x,a) |W_1(y) - W_2(y)| \\
&\leq \gamma ||W_1 - W_2||_\infty \max_a \sum_y p(y|x,a) = \gamma ||W_1 - W_2||_\infty,
\end{aligned}
$$

where in $(a)$ we used $\max_a f(a) - \max_{a'} g(a') \leq \max_a (f(a) - g(a))$. ■

# Exercise: Fixed Point

Revise the Banach fixed point theorem and prove the fixed point property of the Bellman operator.

# Outline

# Question

*Is there any more compact way to describe a value function when we consider an infinite horizon with no discount?*

$\Rightarrow$ *Proper policies and Bellman equations!*

# The Undiscounted Infinite Horizon Setting

The value function is

$$V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{T} r(x_t, \pi(x_t)) | x_0 = x; \pi\Big],$$

where $T$ is the first *random* time when the agent achieves a *terminal* state.

# Proper Policies

## Definition

*A stationary policy* $\pi$ *is proper if* $\exists n \in \mathbb{N}$ *such that* $\forall x \in X$ *the probability of achieving the terminal state* $\bar{x}$ *after n steps is strictly positive. That is*

$$\rho_\pi = \max_x \mathbb{P}(x_n \neq \bar{x} \,|\, x_0 = x, \pi) < 1.$$

# Bounded Value Function

**Proposition**

For any proper policy $\pi$ with parameter $\rho_\pi$ after $n$ steps, the value function is *bounded* as

$$||V^\pi||_\infty \leq r_{\max} \sum_{t \geq 0} \rho_\pi^{\lfloor t/n \rfloor}.$$

# The Undiscounted Infinite Horizon Setting

*Proof.*

By definition of proper policy

$$\mathbb{P}(x_{2n} \neq \bar{x} \,|\, x_0 = x, \pi) = \mathbb{P}(x_{2n} \neq \bar{x} \,|\, x_n \neq \bar{x}, \pi) \times \mathbb{P}(x_n \neq \bar{x} \,|\, x_0 = x, \pi) \leq \rho_\pi^2.$$

Then for any $t \in \mathbb{N}$

$$\mathbb{P}(x_t \neq \bar{x} \,|\, x_0 = x, \pi) \leq \rho_\pi^{\lfloor t/n \rfloor},$$

which implies that *eventually* the terminal state $\bar{x}$ is achieved with probability 1. Then

$$\begin{aligned} ||V^\pi||_\infty &= \max_{x \in X} \mathbb{E}\Big[\sum_{t=0}^{\infty} r(x_t, \pi(x_t)) | x_0 = x; \pi\Big] \\ &\leq r_{\max} \sum_{t>0} \mathbb{P}(x_t \neq \bar{x} \,|\, x_0 = x, \pi) \\ &\leq n r_{\max} + r_{\max} \sum_{t \geq n} \rho_\pi^{\lfloor t/n \rfloor}. \end{aligned}$$

# Bellman Operator

*Assumption.* There exists *at least one proper* policy and for any non-proper policy $\pi$ there exists at least one state $x$ where $V^\pi(x) = -\infty$ (cycles with only negative rewards).

**Proposition ([2])**

Under the previous assumption, the optimal value function is bounded, i.e., $||V^*||_\infty < \infty$ and it is the *unique fixed point* of the *optimal* Bellman operator $\mathcal{T}$ such that for any vector $W \in \mathbb{R}^n$

$$\mathcal{T}W(x) = \max_{a \in A} \big[r(x,a) + \sum_y p(y|x,a)W(y)\big].$$

Furthermore

$$V^* = \lim_{k\to\infty}(\mathcal{T})^k W.$$

# Bellman Operator

## Proposition

Let all the policies $\pi$ be *proper*, then there exist $\mu \in \mathbb{R}^N$ with $\mu > \mathbf{0}$ and a scalar $\beta < 1$ such that, $\forall x, y \in X$, $\forall a \in A$,

$$\sum_y p(y|x,a)\mu(y) \leq \beta\mu(x).$$

Thus both operators $\mathcal{T}$ and $\mathcal{T}^\pi$ are *contraction in the weighted norm* $L_{\infty,\mu}$, that is

$$||\mathcal{T}W_1 - \mathcal{T}W_2||_{\infty,\mu} \leq \beta||W_1 - W_2||_{\infty,\mu}.$$

# Bellman Operator

*Proof.*

Let $\mu$ be the maximum (over policies) of the average time to the termination state. This can be easily casted to a MDP where for any action and any state the rewards are 1 (i.e., for any $x \in X$ and $a \in A$, $r(x, a) = 1$).

Under the assumption that all the policies are proper, then $\mu$ is finite and it is the solution to the dynamic programming equation

$$\mu(x) = 1 + \max_a \sum_y p(y|x, a)\mu(y).$$

Then $\mu(x) \geq 1$ and for any $a \in A$, $\mu(x) \geq 1 + \sum_y p(y|x, a)\mu(y)$. Furthermore,

$$\sum_y p(y|x, a)\mu(y) \leq \mu(x) - 1 \leq \beta\mu(x),$$

for

$$\beta = \max_x \frac{\mu(x) - 1}{\mu(x)} < 1.$$

# Bellman Operator

*Proof (cont'd).*
From this definition of $\mu$ and $\beta$ we obtain the contraction property of $\mathcal{T}$ (similar for $\mathcal{T}^\pi$) in norm $L_{\infty,\mu}$:

$$
\begin{aligned}
||\mathcal{T}W_1 - \mathcal{T}W_2||_{\infty,\mu} &= \max_x \frac{|\mathcal{T}W_1(x) - \mathcal{T}W_2(x)|}{\mu(x)} \\
&\leq \max_{x,a} \frac{\sum_y p(y|x,a)}{\mu(x)} |W_1(y) - W_2(y)| \\
&\leq \max_{x,a} \frac{\sum_y p(y|x,a)\mu(y)}{\mu(x)} \|W_1 - W_2\|_\mu \\
&\leq \beta \|W_1 - W_2\|_\mu
\end{aligned}
$$

■

# Outline

# Question

*How do we compute the value functions / solve an MDP?*

⇒ *Value/Policy Iteration algorithms!*

# System of Equations

The Bellman equation

$$V^{\pi}(x) = r(x, \pi(x)) + \gamma \sum_{y} p(y|x, \pi(x)) V^{\pi}(y).$$

is a *linear* system of equations with $N$ unknowns and $N$ linear constraints.

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The Bellman equation

$$V^\pi(x) = r(x, \pi(x)) + \gamma \sum_y p(y|x, \pi(x)) V^\pi(y).$$

is a *linear* system of equations with $N$ unknowns and $N$ linear constraints.

The optimal Bellman equation

$$V^*(x) = \max_{a \in A} \big[ r(x, a) + \gamma \sum_y p(y|x, a) V^*(y) \big].$$

is a (highly) ***non-linear*** system of equations with $N$ unknowns and $N$ non-linear constraints (i.e., the max operator).

# Value Iteration: the Idea

1. Let $V_0$ be *any* vector in $R^N$

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1. Let $V_0$ be *any* vector in $R^N$
2. At each iteration $k = 1, 2, \ldots, K$
   - Compute $V_{k+1} = \mathcal{T} V_k$
3. Return the *greedy* policy

$$\pi_K(x) \in \arg\max_{a \in A} \Big[ r(x,a) + \gamma \sum_y p(y|x,a) V_K(y) \Big].$$

# Value Iteration: the Guarantees

- From the *fixed point* property of $\mathcal{T}$:

$$\lim_{k \to \infty} V_k = V^*$$

# Value Iteration: the Guarantees

- From the *fixed point* property of $\mathcal{T}$:

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- From the *contraction* property of $\mathcal{T}$

$$||V_{k+1} - V^*||_\infty = ||\mathcal{T} V_k - \mathcal{T} V^*||_\infty \leq \gamma ||V_k - V^*||_\infty \leq \gamma^{k+1} ||V_0 - V^*||_\infty \to 0$$

# Value Iteration: the Guarantees

- From the *fixed point* property of $\mathcal{T}$:

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- From the *contraction* property of $\mathcal{T}$

$$||V_{k+1}-V^*||_\infty = ||\mathcal{T}V_k-\mathcal{T}V^*||_\infty \leq \gamma||V_k-V^*||_\infty \leq \gamma^{k+1}||V_0-V^*||_\infty \to 0$$

- *Convergence rate*. Let $\epsilon > 0$ and $||r||_\infty \leq r_{\max}$, then after *at most*

$$K = \frac{\log(r_{\max}/\epsilon)}{\log(1/\gamma)}$$

iterations $||V_K - V^*||_\infty \leq \epsilon$.

# Value Iteration: the Complexity

One application of the optimal Bellman operator takes $O(N^2|A|)$ operations.

# Value Iteration: Extensions and Implementations

*Q-iteration.*

1. Let $Q_0$ be any Q-function
2. At each iteration $k = 1, 2, \ldots, K$
   - Compute $Q_{k+1} = \mathcal{T} Q_k$
3. Return the greedy policy

$$\pi_K(x) \in \arg\max_{a \in A} Q(x,a)$$

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*Asynchronous VI.*

1. Let $V_0$ be any vector in $R^N$
2. At each iteration $k = 1, 2, \ldots, K$
   - *Choose a state $x_k$*
   - Compute $V_{k+1}(x_k) = \mathcal{T} V_k(x_k)$
3. Return the greedy policy

$$\pi_K(x) \in \arg\max_{a \in A} \left[ r(x,a) + \gamma \sum_y p(y|x,a) V_K(y) \right].$$

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   - *Policy evaluation* given $\pi_k$, compute $V^{\pi_k}$.

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2. At each iteration $k = 1, 2, \ldots, K$
   - *Policy evaluation* given $\pi_k$, compute $V^{\pi_k}$.
   - *Policy improvement*: compute the *greedy* policy

$$\pi_{k+1}(x) \in \arg\max_{a \in A} \Big[ r(x, a) + \gamma \sum_y p(y|x, a) V^{\pi_k}(y) \Big].$$

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3. Return the last policy $\pi_K$

*Remark:* usually $K$ is the smallest $k$ such that $V^{\pi_k} = V^{\pi_{k+1}}$.

# Policy Iteration: the Guarantees

**Proposition**

The policy iteration algorithm generates a sequences of policies with *non-decreasing* performance

$$V^{\pi_{k+1}} \geq V^{\pi_k},$$

and it converges to $\pi^*$ in a *finite* number of iterations.

# Policy Iteration: the Guarantees

*Proof.*

From the definition of the Bellman operators and the greedy policy $\pi_{k+1}$

$$V^{\pi_k} = \mathcal{T}^{\pi_k} V^{\pi_k} \leq \mathcal{T} V^{\pi_k} = \mathcal{T}^{\pi_{k+1}} V^{\pi_k}, \quad (1)$$

# Policy Iteration: the Guarantees

*Proof.*

From the definition of the Bellman operators and the greedy policy $\pi_{k+1}$

$$V^{\pi_k} = \mathcal{T}^{\pi_k} V^{\pi_k} \leq \mathcal{T} V^{\pi_k} = \mathcal{T}^{\pi_{k+1}} V^{\pi_k}, \tag{1}$$

and from the monotonicity property of $\mathcal{T}^{\pi_{k+1}}$, it follows that

$$\begin{aligned} V^{\pi_k} &\leq \mathcal{T}^{\pi_{k+1}} V^{\pi_k}, \\ \mathcal{T}^{\pi_{k+1}} V^{\pi_k} &\leq (\mathcal{T}^{\pi_{k+1}})^2 V^{\pi_k}, \\ &\dots \\ (\mathcal{T}^{\pi_{k+1}})^{n-1} V^{\pi_k} &\leq (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k}, \\ &\dots \end{aligned}$$

# Policy Iteration: the Guarantees

*Proof.*

From the definition of the Bellman operators and the greedy policy $\pi_{k+1}$

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Joining all the inequalities in the chain we obtain

$$V^{\pi_k} \leq \lim_{n\to\infty} (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k} = V^{\pi_{k+1}}.$$

# Policy Iteration: the Guarantees

*Proof.*

From the definition of the Bellman operators and the greedy policy $\pi_{k+1}$

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Joining all the inequalities in the chain we obtain

$$V^{\pi_k} \leq \lim_{n\to\infty} (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k} = V^{\pi_{k+1}}.$$

Then $(V^{\pi_k})_k$ is a non-decreasing sequence.

# Policy Iteration: the Guarantees

*Proof (cont'd).*
Since a finite MDP admits a finite number of policies, then the termination condition is eventually met for a specific $k$.
Thus eq. 1 holds with an equality and we obtain

$$V^{\pi_k} = \mathcal{T} V^{\pi_k}$$

and $V^{\pi_k} = V^*$ which implies that $\pi_k$ is an optimal policy. ■

# *Exercise: Convergence Rate*

*Read the more refined convergence rates in:*
"Improved and Generalized Upper Bounds on the Complexity of Policy Iteration" by B. Scherrer.

# Policy Iteration

*Notation.* For any policy $\pi$ the reward *vector* is $r^\pi(x) = r(x, \pi(x))$ and the transition *matrix* is $[P^\pi]_{x,y} = p(y|x, \pi(x))$

# Policy Iteration: the Policy Evaluation Step

- *Direct computation.* For any policy $\pi$ compute

$$V^\pi = (I - \gamma P^\pi)^{-1} r^\pi.$$

*Complexity:* $O(N^3)$ (improvable to $O(N^{2.807})$).

# Policy Iteration: the Policy Evaluation Step

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*Exercise: prove the previous equality*.

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- *Iterative policy evaluation.* For any policy $\pi$

$$\lim_{n\to\infty} \mathcal{T}^\pi V_0 = V^\pi.$$

*Complexity:* An $\epsilon$-approximation of $V^\pi$ requires $O(N^2 \frac{\log 1/\epsilon}{\log 1/\gamma})$ steps.

# Policy Iteration: the Policy Evaluation Step

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*Exercise: prove the previous equality*.

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$$\lim_{n\to\infty} \mathcal{T}^\pi V_0 = V^\pi.$$

*Complexity:* An $\epsilon$-approximation of $V^\pi$ requires $O(N^2 \frac{\log 1/\epsilon}{\log 1/\gamma})$ steps.

- *Monte-Carlo simulation.* In each state $x$, simulate $n$ trajectories $((x_t^i)_{t\geq 0,})_{1\leq i\leq n}$ following policy $\pi$ and compute

$$\hat{V}^\pi(x) \simeq \frac{1}{n}\sum_{i=1}^{n}\sum_{t\geq 0}\gamma^t r(x_t^i, \pi(x_t^i)).$$

*Complexity:* In each state, the approximation error is $O(1/\sqrt{n})$.

# Policy Iteration: the Policy Improvement Step

- If the policy is evaluated with $V$, then the policy improvement has complexity $O(N|A|)$ (computation of an expectation).

# Policy Iteration: the Policy Improvement Step

- If the policy is evaluated with $V$, then the policy improvement has complexity $O(N|A|)$ (computation of an expectation).
- If the policy is evaluated with $Q$, then the policy improvement has complexity $O(|A|)$ corresponding to

$$\pi_{k+1}(x) \in \arg\max_{a \in A} Q(x, a),$$

# Comparison between Value and Policy Iteration

*Value Iteration*

- *Pros:* each iteration is very *computationally efficient*.
- *Cons:* convergence is only *asymptotic*.

*Policy Iteration*

- *Pros:* converge in a *finite* number of iterations (often small in practice).
- *Cons:* each iteration requires a full *policy evaluation* and it might be expensive.

# Exercise: Review Extensions to Standard DP Algorithms

- Modified Policy Iteration
- $\lambda$-Policy Iteration

# *Exercise: Review Linear Programming*

- Linear Programming: a one-shot approach to computing $V^*$

# Outline

# Things to Remember

- *The Markov Decision Process framework*

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- *State and state-action value function*
- *Bellman equations and Bellman operators*

# Things to Remember

- *The Markov Decision Process framework*
- *The discounted infinite horizon setting*
- *State and state-action value function*
- *Bellman equations and Bellman operators*
- *The value and policy iteration algorithms*

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# Reinforcement Learning

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