The Multi-Arm Bandit Framework

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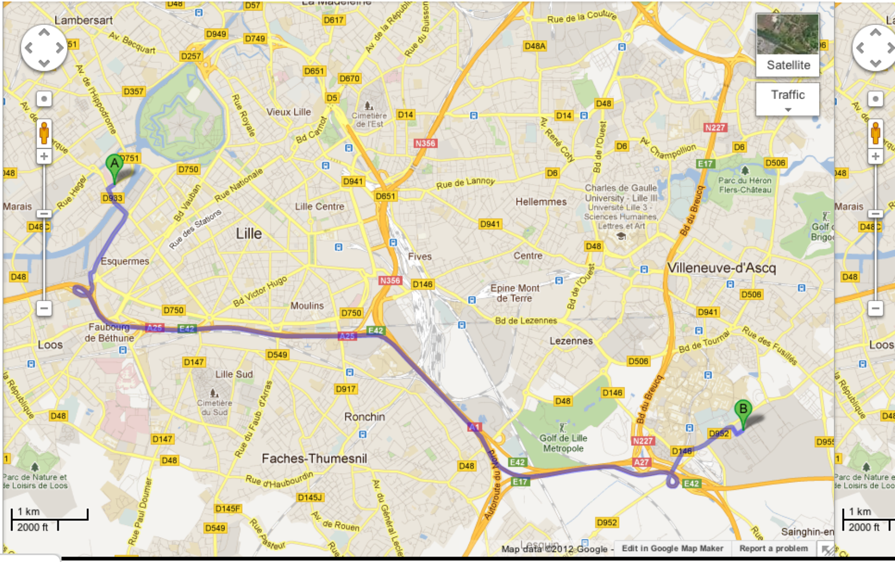
In This Lecture



In This Lecture

Question: which route should we take?

Problem: each day we obtain a ***limited feedback***: traveling time of the *chosen route*

Results: if we do not repeatedly try different options we cannot learn.

Solution: trade off between *optimization* and *learning*.

Outline

Concentration Inequalities

Proposition (Chernoff-Hoeffding Inequality)

Let $X_i \in [a_i, b_i]$ be n *independent* r.v. with mean $\mu_i = \mathbb{E}X_i$. Then

$$\mathbb{P}\Big[\Big|\sum_{i=1}^{n}(X_i - \mu_i)\Big| \geq \epsilon\Big] \leq 2\exp\Big(-\frac{2\epsilon^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\Big).$$

Concentration Inequalities

Proof.

$$
\begin{aligned}
\mathbb{P}\Big(\sum_{i=1}^n X_i - \mu_i \geq \epsilon\Big) &= \mathbb{P}(e^{s\sum_{i=1}^n X_i-\mu_i} \geq e^{s\epsilon}) \\
&\leq e^{-s\epsilon}\mathbb{E}[e^{s\sum_{i=1}^n X_i-\mu_i}], \quad \text{Markov inequality} \\
&= e^{-s\epsilon}\prod_{i=1}^n \mathbb{E}[e^{s(X_i-\mu_i)}], \quad \text{independent random variables} \\
&\leq e^{-s\epsilon}\prod_{i=1}^n e^{s^2(b_i-a_i)^2/8}, \quad \text{Hoeffding inequality} \\
&= e^{-s\epsilon+s^2\sum_{i=1}^n(b_i-a_i)^2/8}
\end{aligned}
$$

If we choose $s = 4\epsilon/\sum_{i=1}^n(b_i - a_i)^2$, the result follows.
Similar arguments hold for $\mathbb{P}\big(\sum_{i=1}^n X_i - \mu_i \leq -\epsilon\big)$.

Concentration Inequalities

Finite sample guarantee:

$$\mathbb{P}\Bigg[\underbrace{\Big|\frac{1}{n}\sum_{t=1}^{n} X_t - \mathbb{E}[X_1]\Big|}_{\text{deviation}} > \underbrace{\epsilon}_{\text{accuracy}}\Bigg] \leq \underbrace{2\exp\Big(-\frac{2n\epsilon^2}{(b-a)^2}\Big)}_{\text{confidence}}$$

Concentration Inequalities

Finite sample guarantee:

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{t=1}^{n} X_t - \mathbb{E}[X_1]\right| > (b-a)\sqrt{\frac{\log 2/\delta}{2n}}\right] \leq \delta$$

Concentration Inequalities

Finite sample guarantee:

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{t=1}^{n} X_t - \mathbb{E}[X_1]\right| > \epsilon\right] \leq \delta$$

if $n \geq \frac{(b-a)^2 \log 2/\delta}{2\epsilon^2}$.

Outline

The Multi–armed Bandit Game

The learner has $i = 1, \ldots, N$ arms (options, experts, ...)

At each round $t = 1, \ldots, n$

- At the same time
  - The environment chooses a vector of *rewards* $\{X_{i,t}\}_{i=1}^{N}$
  - The learner chooses an arm I_t
- The learner receives a reward $X_{I_t,t}$
- The environment ***does not*** reveal the rewards of the other arms

The Multi–armed Bandit Game (cont'd)

The regret

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} \mathbb{E}\Big[\sum_{t=1}^{n} X_{i,t}\Big] - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big]$$

The expectation summarizes any possible source of randomness (either in X or in the algorithm)

The Exploration–Exploitation Lemma

Problem 1: The environment ***does not*** reveal the rewards of the arms not pulled by the learner
$\Rightarrow$ the learner should *gain information* by repeatedly pulling all the arms $\Rightarrow$ ***exploration***

Problem 2: Whenever the learner pulls a ***bad arm***, it suffers some regret
$\Rightarrow$ the learner should *reduce the regret* by repeatedly pulling the best arm $\Rightarrow$ ***exploitation***
Challenge: The learner should solve two opposite problems!
Challenge: The learner should solve the *exploration-exploitation* dilemma!

The Multi–armed Bandit Game (cont'd)

Examples

- Packet routing
- Clinical trials
- Web advertising
- Computer games
- Resource mining
- ...

Outline

The Stochastic Multi–armed Bandit Problem

Definition

The environment is stochastic

- *Each arm has a distribution* ν_i *bounded in* $[0, 1]$ *and characterized by an expected value* μ_i
- *The rewards are i.i.d.* $X_{i,t} \sim \nu_i$

The Stochastic Multi–armed Bandit Problem (cont'd)

Notation

- Number of times arm i has been pulled after n rounds

$$T_{i,n} = \sum_{t=1}^{n} \mathbb{I}\{I_t = i\}$$

- Regret

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} \mathbb{E}\Big[\sum_{t=1}^{n} X_{i,t}\Big] - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big]$$

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} (n\mu_i) - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big]$$

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} (n\mu_i) - \sum_{i=1}^{N} \mathbb{E}[T_{i,n}]\mu_i$$

$$R_n(\mathcal{A}) = n\mu_{i^*} - \sum^{N} \mathbb{E}[T_{i,n}]\mu_i$$

The Stochastic Multi–armed Bandit Problem (cont'd)

$$R_n(\mathcal{A}) = \sum_{i \neq i^*} \mathbb{E}[T_{i,n}]\Delta_i$$

$\Rightarrow$ we only need to study the *expected number of pulls* of the *suboptimal* arms

The Stochastic Multi–armed Bandit Problem (cont'd)

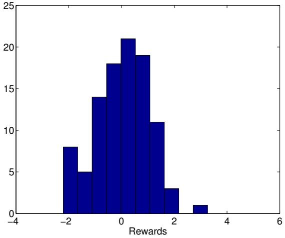
Optimism in Face of Uncertainty Learning (OFUL)

Whenever we are *uncertain* about the outcome of an arm, we consider the *best possible world* and choose the *best arm*.

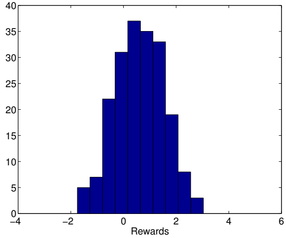
Why it works:

- If the *best possible world* is correct ⇒ *no regret*
- If the *best possible world* is wrong ⇒ *the reduction in the uncertainty is maximized*

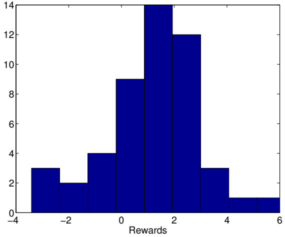
The Stochastic Multi–armed Bandit Problem (cont'd)



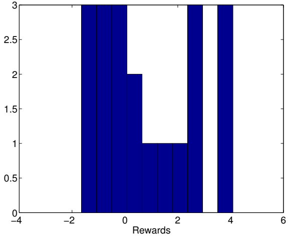
pulls $=$ 100



pulls $=$ 200



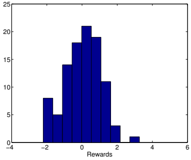
pulls $=$ 50

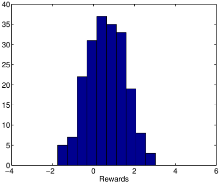


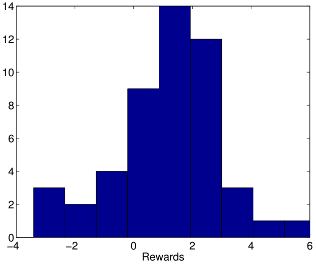
pulls $=$ 20

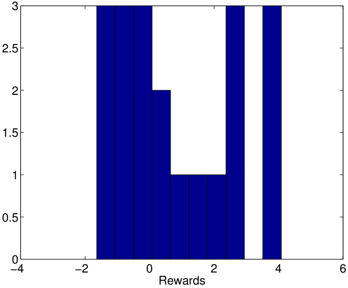
The Stochastic Multi–armed Bandit Problem (cont'd)

Optimism in face of uncertainty



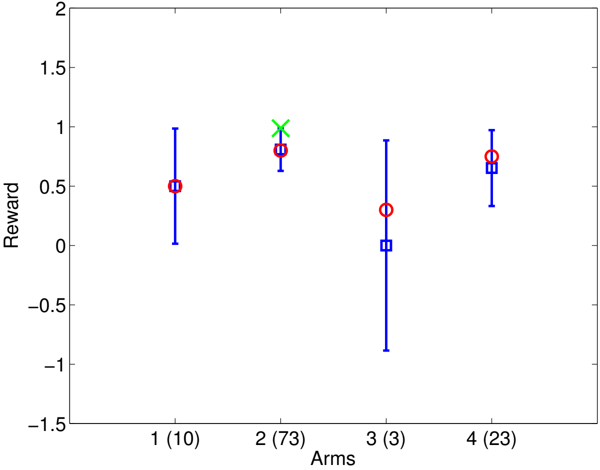






The Upper–Confidence Bound (UCB) Algorithm

The idea



The Upper–Confidence Bound (UCB) Algorithm

Show time!

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

At each round $t = 1, \ldots, n$

- Compute the *score* of each arm i

$$B_i = (\textit{optimistic} \text{ score of arm } i)$$

- Pull arm

$$I_t = \arg\max_{i=1,\ldots,N} B_{i,s,t}$$

- Update the number of pulls $T_{I_t,t} = T_{I_t,t-1} + 1$

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

The score (with parameters ρ and δ)

$$B_i = (\textit{optimistic} \text{ score of arm } i)$$

$$B_{i,s,t} = (\textit{optimistic} \text{ score of arm } i \text{ if pulled } s \text{ times up to round } t)$$

$$B_{i,s,t} = \text{knowledge} \underbrace{+}_{\textit{optimism}} \text{uncertainty}$$

$$B_{i,s,t} = \hat{\mu}_{i,s} + \rho\sqrt{\frac{\log 1/\delta}{2s}}$$

Optimism in face of uncertainty:
Current knowledge: average rewards $\hat{\mu}_{i,s}$
Current uncertainty: number of pulls s

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

Do you remember Chernoff-Hoeffding?

Theorem

Let $X_1, \ldots, X_n$ be i.i.d. samples from a distribution bounded in $[a, b]$, then for any $\delta \in (0, 1)$

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{t=1}^{n} X_t - \mathbb{E}[X_1]\right| > (b-a)\sqrt{\frac{\log 2/\delta}{2n}}\right] \leq \delta$$

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

After s pulls, arm i

$$\mathbb{P}\left[\mathbb{E}[X_i] \leq \frac{1}{s}\sum_{t=1}^{s} X_{i,t} + \sqrt{\frac{\log 1/\delta}{2s}}\right] \geq 1-\delta$$

$$\mathbb{P}\left[\mu_i \leq \hat{\mu}_{i,s} + \sqrt{\frac{\log 1/\delta}{2s}}\right] \geq 1-\delta$$

$\Rightarrow$ UCB uses an *upper confidence bound* on the expectation

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

Theorem

For any set of N arms with distributions bounded in $[0, b]$, if $\delta = 1/t$, then UCB(ρ) with $\rho > 1$, achieves a regret

$$R_n(\mathcal{A}) \leq \sum_{i \neq i^*} \left[\frac{4b^2}{\Delta_i} \rho \log(n) + \Delta_i \left(\frac{3}{2} + \frac{1}{2(\rho - 1)} \right) \right]$$

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

Let $N = 2$ with $i^* = 1$

$$R_n(\mathcal{A}) \leq O\left(\frac{1}{\Delta}\rho \log(n)\right)$$

Remark 1: the *cumulative* regret slowly increases as $\log(n)$
Remark 2: the *smaller the gap* the *bigger the regret*... why?

The Upper–Confidence Bound (UCB) Algorithm (cont'd)

Show time (again)!

The Worst–case Performance

Remark: the regret bound is *distribution–dependent*

$$R_n(\mathcal{A}; \Delta) \leq O\left(\frac{1}{\Delta}\rho \log(n)\right)$$

Meaning: the algorithm is able to *adapt to the specific problem* at hand!

Worst–case performance: what is the distribution which leads to the worst possible performance of UCB? what is the distribution–free performance of UCB?

$$R_n(\mathcal{A}) = \sup_{\Delta} R_n(\mathcal{A}; \Delta)$$

The Worst–case Performance

Problem: it seems like if $\Delta \to 0$ then the regret tends to infinity...
... nosense because the regret is defined as

$$R_n(\mathcal{A}; \Delta) = \mathbb{E}[T_{2,n}]\Delta$$

then if Δ_i is small, the regret is also small...
In fact

$$R_n(\mathcal{A}; \Delta) = \min\left\{ O\left(\frac{1}{\Delta}\rho\log(n)\right), \mathbb{E}[T_{2,n}]\Delta \right\}$$

The Worst–case Performance

Then

$$R_n(\mathcal{A}) = \sup_{\Delta} R_n(\mathcal{A}; \Delta) = \sup_{\Delta} \min \left\{ O\left(\frac{1}{\Delta}\rho \log(n)\right), n\Delta \right\} \approx \sqrt{n}$$

for $\Delta = \sqrt{1/n}$

Tuning the confidence δ of UCB

Remark: UCB is an *anytime* algorithm ($\delta = 1/t$)

$$B_{i,s,t} = \hat{\mu}_{i,s} + \rho\sqrt{\frac{\log t}{2s}}$$

Remark: If the time horizon n is known then the optimal choice is $\delta = 1/n$

$$B_{i,s,t} = \hat{\mu}_{i,s} + \rho\sqrt{\frac{\log n}{2s}}$$

Tuning the confidence δ of UCB (cont'd)

Intuition: UCB should pull the suboptimal arms

- *Enough*: so as to understand which arm is the best
- *Not too much*: so as to keep the regret as small as possible

The confidence $1-\delta$ has the following impact (similar for ρ)

- *Big* $1-\delta$: high level of *exploration*
- *Small* $1-\delta$: high level of *exploitation*

Solution: depending on the time horizon, we can tune how to trade-off between exploration and exploitation

Tuning the confidence δ of UCB (cont'd)

Let's dig into the (1 page and half!!) proof.

Define the (high-probability) event *[statistics]*

$$\mathcal{E} = \left\{ \forall i, s \quad \left| \hat{\mu}_{i,s} - \mu_i \right| \leq \sqrt{\frac{\log 1/\delta}{2s}} \right\}$$

By Chernoff-Hoeffding $\mathbb{P}[\mathcal{E}] \geq 1 - nN\delta$.
At time t we pull arm i *[algorithm]*

$$B_{i,T_{i,t-1}} \geq B_{i^*,T_{i^*,t-1}}$$

$$\hat{\mu}_{i,T_{i,t-1}} + \sqrt{\frac{\log 1/\delta}{2T_{i,t-1}}} \geq \hat{\mu}_{i^*,T_{i^*,t-1}} + \sqrt{\frac{\log 1/\delta}{2T_{i^*,t-1}}}$$

On the event $\mathcal{E}$ we have *[math]*

$$\mu_i + 2\sqrt{\frac{\log 1/\delta}{2T_{i,t-1}}} \geq \mu_{i^*}$$

Tuning the confidence δ of UCB (cont'd)

Assume t is the last time i is pulled, then $T_{i,n} = T_{i,t-1} + 1$, thus

$$\mu_i + 2\sqrt{\frac{\log 1/\delta}{2(T_{i,n} - 1)}} \geq \mu_{i^*}$$

Reordering *[math]*

$$T_{i,n} \leq \frac{\log 1/\delta}{2\Delta_i^2} + 1$$

under event $\mathcal{E}$ and thus with probability $1 - nN\delta$.
Moving to the expectation *[statistics]*

$$\mathbb{E}[T_{i,n}] = \mathbb{E}[T_{i,n}\mathbb{I}\mathcal{E}] + \mathbb{E}[T_{i,n}\mathbb{I}\mathcal{E}^C]$$
$$\mathbb{E}[T_{i,n}] \leq \frac{\log 1/\delta}{2\Delta_i^2} + 1 + n(nN\delta)$$

Trading-off the two terms $\delta = 1/n^2$, we obtain

$$\hat{\mu}_{i,T_{i,t-1}} + \sqrt{\frac{2\log n}{2T}}$$

Tuning the confidence δ of UCB (cont'd)

Trading-off the two terms $\delta = 1/n^2$, we obtain

$$\hat{\mu}_{i,T_{i,t-1}} + \sqrt{\frac{2\log n}{2T_{i,t-1}}}$$

and

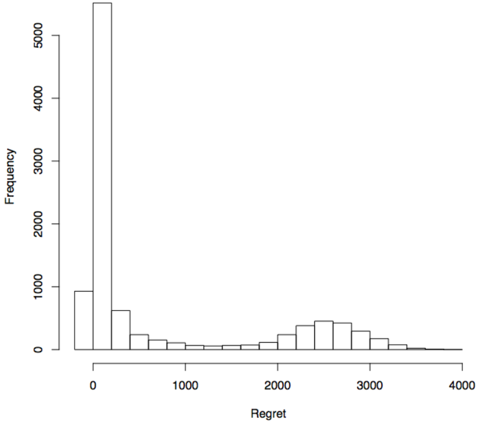
$$\mathbb{E}[T_{i,n}] \leq \frac{\log n}{\Delta_i^2} + 1 + N$$

Tuning the confidence δ of UCB (cont'd)

Multi–armed Bandit: the same for $\delta = 1/t$ and $\delta = 1/n$...
... **almost** (i.e., in expectation)

Tuning the confidence δ of UCB (cont'd)

The value–at–risk of the regret for UCB-anytime



Tuning the ρ of UCB (cont'd)

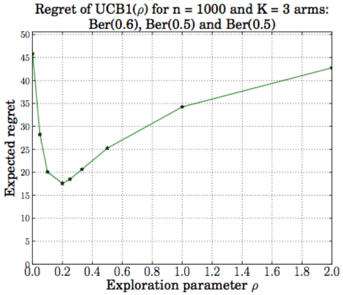
UCB values (for the $\delta = 1/n$ algorithm)

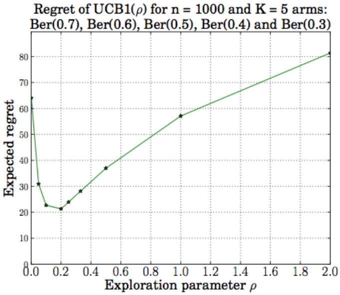
$$B_{i,s} = \hat{\mu}_{i,s} + \rho\sqrt{\frac{\log n}{2s}}$$

Theory

- $\rho < 0.5$, polynomial regret w.r.t. n
- $\rho > 0.5$, logarithmic regret w.r.t. n

Practice: $\rho = 0.2$ is often the best choice





Improvements over UCB: UCB-V

Idea: use Bernstein bounds with empirical variance

Algorithm:

$$B_{i,s,t} = \hat{\mu}_{i,s} + \sqrt{\frac{\log t}{2s}}$$

$$B^{V}_{i,s,t} = \hat{\mu}_{i,s} + \sqrt{\frac{2\hat{\sigma}^2_{i,s} \log t}{s}} + \frac{8 \log t}{3s}$$

$$R_n \leq O\Big(\frac{1}{\Delta} \log n\Big)$$

$$R_n \leq O\Big(\frac{\sigma^2}{\Delta} \log n\Big)$$

Improvements over UCB: KL-UCB

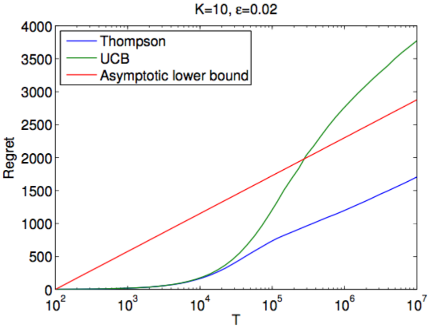
Idea: use Kullback–Leibler bounds which are tighter than other bounds

Algorithm: the algorithm is still index–based but a bit more complicated

$$R_n \leq O\Big(\frac{1}{\Delta} \log n\Big) \qquad\qquad R_n \leq O\Big(\frac{1}{KL(\nu, \nu_{i^*})} \log n\Big)$$

Improvements over UCB: Thompson strategy

Idea: Keep a distribution over the possible values of μ_i
Algorithm: Bayesian approach. Compute the posterior distributions given the samples.



Back to UCB: the Lower Bound

Theorem

For any stochastic bandit $\{\nu_i\}$, any algorithm $\mathcal{A}$ has a regret

$$\lim_{n\to\infty} \frac{R_n}{\log n} \geq \frac{\Delta_i}{\inf_\nu KL(\nu_i, \nu)}$$

Problem: this is just asymptotic

Open Question: what is the finite-time lower bound?

Outline

The Non–Stochastic Multi–armed Bandit Problem

Definition

The environment is *adversarial*

- Arms have no fixed distribution
- The rewards $X_{i,t}$ are arbitrarily chosen by the environment

The Non–Stochastic Multi–armed Bandit Problem (cont'd)

The (non–stochastic bandit) regret

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} \mathbb{E}\Big[\sum_{t=1}^{n} X_{i,t}\Big] - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big]$$

$$R_n(\mathcal{A}) = \max_{i=1,\dots,N} \sum_{t=1}^{n} X_{i,t} - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big]$$

The Exponentially Weighted Average Forecaster

Initialize the weights $w_{i,0} = 1$

- Compute $(W_{t-1} = \sum_{i=1}^{N} w_{i,t-1})$

$$\hat{p}_{i,t} = \frac{w_{i,t-1}}{W_{t-1}}$$

- Choose the arm at random

$$I_t \sim \hat{\mathbf{p}}_t$$

- Observe the rewards $\{X_{i,t}\}$
- Receive a reward $X_{I_t,t}$
- Update

$$w_{i,t} = w_{i,t-1} \exp\big(+ \eta X_{i_t,t}\big)$$

The Non–Stochastic Multi–armed Bandit Problem (cont'd)

Problem: we only observe the reward of the specific arm chosen at time t!! (i.e., only $X_{I_t,t}$ is observed)

The Exponentially Weighted Average Forecaster

Initialize the weights $w_{i,0} = 1$

- Compute $(W_{t-1} = \sum_{i=1}^{N} w_{i,t-1})$

$$\hat{p}_{i,t} = \frac{w_{i,t-1}}{W_{t-1}}$$

- Choose the arm at random

$$I_t \sim \hat{\mathbf{p}}_t$$

- ~~Observe the rewards $\{X_{i,t}\}$~~
- Receive a reward $X_{I_t,t}$
- Update

$$w_{i,t} = w_{i,t-1} \exp\left(\eta X_{i_t,t}\right) \Rightarrow \text{this update is not possible}$$

The Non–Stochastic Multi–armed Bandit Problem (cont'd)

We use the *importance weight* trick

$$\hat{X}_{i,t} = \begin{cases} \frac{X_{i,t}}{\hat{p}_{i,t}} & \text{if } i = I_t \\ 0 & \text{otherwise} \end{cases}$$

Why it is a good idea:

$$\mathbb{E}\big[\hat{X}_{i,t}\big] = \frac{X_{i,t}}{\hat{p}_{i,t}}\hat{p}_{i,t} + 0(1 - \hat{p}_{i,t}) = X_{i,t}$$

$\hat{X}_{i,t}$ is an *unbiased* estimator of $X_{i,t}$

The Exp3 Algorithm

Exp3: Exponential-weight algorithm for Exploration and Exploitation

Initialize the weights $w_{i,0} = 1$

- Compute $\left(W_{t-1} = \sum_{i=1}^{N} w_{i,t-1}\right)$

$$\hat{p}_{i,t} = \frac{w_{i,t-1}}{W_{t-1}}$$

- Choose the arm at random

$$I_t \sim \hat{\mathbf{p}}_t$$

- Receive a reward $X_{I_t,t}$
- Update

$$w_{i,t} = w_{i,t-1} \exp\left(\eta \hat{X}_{i_t,t}\right)$$

The Exp3 Algorithm

Question: is this enough? is this algorithm actually exploring enough?
Answer: more or less...

- Exp3 has a small regret *in expectation*
- Exp3 might have large deviations with *high probability* (ie, from time to time it may *concentrate* $\hat{\mathbf{p}}_t$ *on the wrong arm* for too long and then incur a large regret)

The Exp3 Algorithm

Fix: add some extra uniform exploration

Initialize the weights $w_{i,0} = 1$

- Compute ($W_{t-1} = \sum_{i=1}^{N} w_{i,t-1}$)

$$\hat{p}_{i,t} = (1-\gamma)\frac{w_{i,t-1}}{W_{t-1}} + \frac{\gamma}{K}$$

- Choose the arm at random

$$I_t \sim \hat{\mathbf{p}}_t$$

- Receive a reward $X_{I_t,t}$
- Update

$$w_{i,t} = w_{i,t-1} \exp\left(\eta \hat{X}_{i_t,t}\right)$$

The Exp3 Algorithm

Theorem

If Exp3 is run with $\gamma = \eta$, then it achieves a regret

$$R_n(\mathcal{A}) = \max_{i=1,\ldots,N} \sum_{t=1}^{n} X_{i,t} - \mathbb{E}\Big[\sum_{t=1}^{n} X_{I_t,t}\Big] \leq (e-1)\gamma G_{\max} + \frac{N \log N}{\gamma}$$

with $G_{\max} = \max_{i=1,\ldots,N} \sum_{t=1}^{n} X_{i,t}$.

The Exp3 Algorithm

Theorem

If Exp3 is run with

$$\gamma = \eta = \sqrt{\frac{N \log N}{(e-1)n}}$$

then it achieves a regret

$$R_n(\mathcal{A}) \leq O(\sqrt{nN \log N})$$

The Exp3 Algorithm

Comparison with online learning

$$R_n(Exp3) \leq O(\sqrt{nN \log N})$$

$$R_n(EWA) \leq O(\sqrt{n \log N})$$

Intuition: in online learning at each round we obtain N feedbacks, while in bandits we receive 1 feedback.

The Improved-Exp3 Algorithm

Initialize the weights $w_{i,0} = 1$

- Compute $(W_{t-1} = \sum_{i=1}^{N} w_{i,t-1})$

$$\hat{p}_{i,t} = (1-\gamma)\frac{w_{i,t-1}}{W_{t-1}} + \frac{\gamma}{K}$$

- Choose the arm at random

$$I_t \sim \hat{\mathbf{p}}_t$$

- Receive a reward $X_{I_t,t}$
- Compute

$$\widetilde{X}_{i,t} = \hat{X}_{i,t} + \frac{\beta}{\hat{p}_{i,t}}$$

- Update

$$w_{i,t} = w_{i,t-1} \exp\left(\eta \widetilde{X}_{i_t,t}\right)$$

The Improved-Exp3 Algorithm

Theorem

If Improved-Exp3 is run with parameters in the ranges

$$\gamma \leq \frac{1}{2}; \quad 0 \leq \eta \leq \frac{\gamma}{2N}; \quad \sqrt{\frac{1}{nN} \log \frac{N}{\delta}} \leq \beta \leq 1$$

then it achieves a regret

$$R_n^{HP}(\mathcal{A}) \leq n\big(\gamma + \eta(1+\beta)N\big) + \frac{\log N}{\eta} + 2nN\beta$$

with probability at least $1 - \delta$.

The Improved-Exp3 Algorithm

Theorem

If Improved-Exp3 is run with parameters in the ranges

$$\beta = \sqrt{\frac{1}{nN}\log\frac{N}{\delta}}; \quad \gamma = \frac{4N\beta}{3+\beta}; \quad \eta = \frac{\gamma}{2N}$$

then it achieves a regret

$$R_n^{HP}(\mathcal{A}) \leq \frac{11}{2}\sqrt{nN\log(N/\delta)} + \frac{\log N}{2}$$

with probability at least $1-\delta$*.*

Outline

Repeated Two–Player Zero–Sum Games

A two–player zero–sum game

	A	*B*	*C*
1	*30*, *-30*	*-10*, *10*	*20*, *-20*
2	*10*, *-10*	*-20*, *20*	*-20*, *20*

Nash equilibrium:
A set of strategies is a Nash equilibrium if *no player* can do better by *unilaterally changing* his strategy.
Red: take action *1* with *prob. 4/7* and action *2* with *prob. 3/7*
Blue: take action *A* with *prob. 0*, action *B* with *prob. 4/7*, and action *C* with *prob. 3/7*
Value of the game: $V = 20/7$ (reward of Red at the equilibrium)

Repeated Two–Player Zero–Sum Games

At each round t

- Row player computes a mixed strategy $\hat{\mathbf{p}}_t = (\hat{p}_{1,t}, \dots, \hat{p}_{N,t})$
- Column player computes a mixed strategy $\hat{\mathbf{q}}_t = (\hat{q}_{1,t}, \dots, \hat{q}_{M,t})$
- Row player selects action $I_t \in \{1, \dots, N\}$
- Column player selects action $J_t \in \{1, \dots, M\}$
- Row player suffers $\ell(I_t, J_t)$
- Column player suffers $-\ell(I_t, J_t)$

Value of the game

$$V = \max_{\mathbf{q}} \min_{\mathbf{p}} \overline{\ell}(\mathbf{p}, \mathbf{q})$$

with

$$\overline{\ell}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_i q_j \ell(i, j)$$

Repeated Two–Player Zero–Sum Games

Question: what if the two players are both bandit algorithms (e.g., Exp3)?

Row player: a bandit algorithm is able to minimize

$$R_n(\text{row}) = \sum_{t=1}^{n} \ell_{I_t,J_t} - \min_{i=1,\dots,N} \sum_{t=1}^{n} \ell_{i,J_t}$$

Col player: a bandit algorithm is able to minimize

$$R_n(\text{col}) = \sum_{t=1}^{n} \ell_{I_t,J_t} - \min_{j=1,\dots,M} \sum_{t=1}^{n} \ell_{I_t,j}$$

Repeated Two–Player Zero–Sum Games

Theorem

If both the row and column players play according to an Hannan-consistent strategy, then

$$\lim\sup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = V$$

Repeated Two–Player Zero–Sum Games

Theorem

The empirical distribution of plays

$$\hat{p}_{i,n} = \frac{1}{n}\sum_{t=1}^{n}\mathbb{I}\{I_t = i\} \quad \hat{q}_{j,n} = \frac{1}{n}\sum_{t=1}^{n}\mathbb{I}\{J_t = j\}$$

induces a product distribution $\hat{\mathbf{p}}_n \times \hat{\mathbf{q}}_n$ which converges to the set of Nash equilibria $\mathbf{p} \times \mathbf{q}$.

Repeated Two–Player Zero–Sum Games

Proof idea.

Since $\bar{\ell}(\mathbf{p}, J_t)$ is linear, over the simplex, the minimum is at one of the corners *[math]*

$$\min_{i=1,\dots,N} \frac{1}{N} \sum_{t=1}^{n} \ell(i, J_t) = \min_{\mathbf{p}} \frac{1}{n} \sum_{t=1}^{n} \bar{\ell}(\mathbf{p}, J_t)$$

We consider the empirical probability of the row player *[def]*

$$\hat{q}_{j,n} = \frac{1}{n} \sum_{t=1}^{n} \mathbb{I} J_t = j$$

Elaborating on it *[math]*

$$\begin{aligned} \min_{\mathbf{p}} \frac{1}{n} \sum_{t=1}^{n} \bar{\ell}(\mathbf{p}, J_t) &= \min_{\mathbf{p}} \sum_{j=1}^{M} \hat{q}_{j,n} \bar{\ell}(\mathbf{p}, j) \\ &= \min_{\mathbf{p}} \bar{\ell}(\mathbf{p}, \hat{\mathbf{q}}_n) \\ &\leq \max_{\mathbf{q}} \min_{\mathbf{p}} \bar{\ell}(\mathbf{p}, \mathbf{q}) = V \end{aligned}$$

Repeated Two–Player Zero–Sum Games

Proof idea.
By definition of Hannan's consistent strategy *[def]*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = \min_{i=1,\dots,N} \frac{1}{n}\sum_{t=1}^{n} \ell(i, J_t)$$

Then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) \leq V$$

If we do the same for the other player *[zero–sum game]*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) \geq V$$

Repeated Two–Player Zero–Sum Games

Question: how fast do they converge to the Nash equilibrium?
Answer: it depends on the specific algorithm. For EWA(η), we now that

$$\sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1,\dots,N} \sum_{t=1}^{n} \ell(i, J_t) \leq \frac{\log N}{\eta} + \frac{n\eta}{8} + \sqrt{\frac{n}{2} \log \frac{1}{\delta}}$$

Repeated Two–Player Zero–Sum Games

Generality of the results

- Players do not know the payoff matrix
- Players do not observe the loss of the other player
- Players do not even observe the action of the other player

Internal Regret and Correlated Equilibria

External (expected) regret

$$R_n = \sum_{t=1}^{n} \bar{\ell}(\hat{\mathbf{p}}_t, y_t) - \min_{i=1,\dots,N} \sum_{t=1}^{n} \ell(i, y_t)$$

$$= \max_{i=1,\dots,N} \sum_{t=1}^{n} \sum_{j=1}^{N} \hat{p}_{j,t} \big(\ell(j, y_t) - \ell(i, y_t)\big)$$

Internal (expected) regret

$$R_n^I = \max_{i,j=1,\dots,N} \sum_{t=1}^{n} \hat{p}_{j,t} \big(\ell(i, y_t) - \ell(j, y_t)\big)$$

Internal Regret and Correlated Equilibria

Internal (expected) regret

$$R_n^I = \max_{i,j=1,\dots,N} \sum_{t=1}^{n} \hat{p}_{j,t} \big(\ell(i, y_t) - \ell(j, y_t) \big)$$

Intuition: an algorithm has *small internal regret* if, for each pair of experts (i, j), the learner does not regret of not having followed expert j each time it followed expert i.

Internal Regret and Correlated Equilibria

Theorem

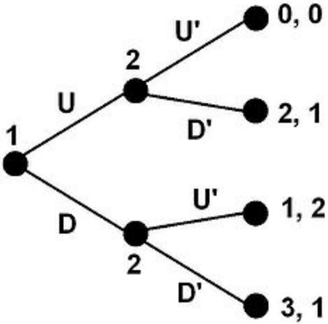
Given a K–person game with a set of correlated equilibria $\mathcal{C}$. If all the players are internal–regret minimizers, then the distance between the empirical distribution of plays and the set of correlated equilibria $\mathcal{C}$ converges to 0.

Nash Equilibria in Extensive Form Games

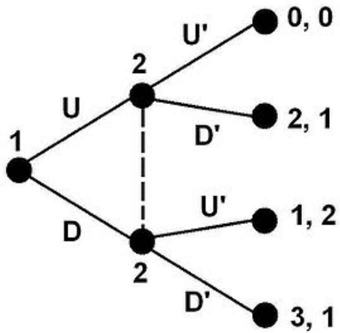
A powerful model for *sequential* games

- Checkers / Chess / Go
- Poker
- Bargaining
- Monitoring
- Patrolling
- ...

Nash Equilibria in Extensive Form Games



Nash Equilibria in Extensive Form Games



Nash Equilibria in Extensive Form Games

No details about the algorithm... but...

Theorem

If player k selects actions according to the counterfactual regret minimization algorithm, then it achieves a regret

$$R_{k,T} \leq \#\, states \sqrt{\frac{\#\, actions}{T}}$$

Theorem

In a two–player zero–sum extensive form game, counterfactual regret minimization algorithms achieves an 2ϵ-Nash equilibrium, with

$$\epsilon \leq \#\, states \sqrt{\frac{\#\, actions}{T}}$$

Outline

The Best Arm Identification Problem

Motivating Examples

- Find the best shortest path in a limited number of days
- Maximize the confidence about the best treatment after a finite number of patients
- Discover the best advertisements after a training phase
- ...

The Best Arm Identification Problem

Objective: given a fixed budget n, return the best arm $i^* = \arg\max_i \mu_i$ at the end of the experiment
Measure of performance: the probability of error

$$\mathbb{P}[J_n \neq i^*]$$

$$\mathbb{P}[J_n \neq i^*] \leq \sum_{i=1}^{N} \exp\big(- T_{i,n}\Delta_i^2\big)$$

Algorithm idea: mimic the behavior of the optimal strategy

$$T_{i,n} = \frac{\frac{1}{\Delta_i^2}}{\sum_{j=1}^{N} \frac{1}{\Delta_j^2}} n$$

The Best Arm Identification Problem

The Successive Reject Algorithm

- Divide the budget in $N-1$ phases. Define $(\overline{\log}(N) = 0.5 + \sum_{i=2}^{N} 1/i)$

$$n_k = \frac{1}{\overline{\log} K} \frac{n-N}{N+1-k}$$

- Set of active arms A_k at phase k ($A_1 = \{1, \ldots, N\}$)
- For each phase $k = 1, \ldots, N-1$
  - For each arm $i \in A_k$, pull arm i for $n_k - n_{k-1}$ rounds
  - Remove the worst arm

$$A_{k+1} = A_k \setminus \arg\min_{i \in A_k} \hat{\mu}_{i,n_k}$$

- Return the only remaining arm $J_n = A_N$

The Best Arm Identification Problem

The Successive Reject Algorithm

Theorem

The successive reject algorithm have a probability of doing a mistake of

$$\mathbb{P}[J_n \neq i^*] \leq \frac{K(K-1)}{2} \exp\Big(-\frac{n-N}{\overline{\log} N H_2}\Big)$$

with $H_2 = \max_{i=1,\ldots,N} i\Delta_{(i)}^{-2}$.

The Best Arm Identification Problem

The UCB-E Algorithm

- Define an exploration parameter a
- Compute

$$B_{i,s} = \hat{\mu}_{i,s} + \sqrt{\frac{a}{s}}$$

- Select

$$I_t = \arg\max_{B_{i,s}}$$

- At the end return

$$J_n = \arg\max_i \hat{\mu}_{i,T_{i,n}}$$

The Best Arm Identification Problem

The UCB-E Algorithm

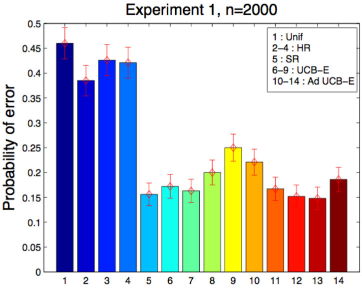
Theorem

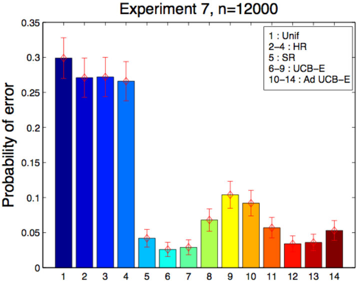
The UCB-E algorithm with $a = \frac{25}{36}\frac{n-N}{H_1}$ *has a probability of doing a mistake of*

$$\mathbb{P}[J_n \neq i^*] \leq 2nN \exp\Big(-\frac{2a}{25}\Big)$$

with $H_1 = \sum_{i=1}^N 1/\Delta_i^2$.

The Best Arm Identification Problem





The Active Bandit Problem

Motivating Examples

- N production lines
- The test of the performance of a line is expensive
- We want an accurate estimation of the performance of each production line

The Active Bandit Problem

Objective: given a fixed budget n, return the an estimate of the means $\hat{\mu}_{i,t}$ which is as accurate as possible for all the arms

Notice: Given an arm has a mean μ_i and a variance σ_i^2, if it is pulled $T_{i,n}$ times, then

$$L_{i,n} = \mathbb{E}\big[(\hat{\mu}_{i,T_{i,n}} - \mu_i)^2\big] = \frac{\sigma_i^2}{T_{i,n}}$$

$$L_n = \max_i L_{i,n}$$

The Active Bandit Problem

Problem: what are the number of pulls $(T_{1,n}, \ldots, T_{N,n})$ (such that $\sum T_{i,n} = n$) which minimizes the loss?

$$(T^*_{1,n}, \ldots, T^*_{N,n}) = \arg \min_{(T_{1,n},\ldots,T_{N,n})} L_n$$

Answer

$$T^*_{i,n} = \frac{\sigma_i^2}{\sum_{j=1}^N \sigma_j^2} n$$

$$L^*_n = \frac{\sum_{i=1}^N \sigma_i^2}{n} = \frac{\Sigma}{n}$$

The Active Bandit Problem

Objective: given a fixed budget n, return the an estimate of the means $\hat{\mu}_{i,t}$ which is as accurate as possible for all the arms
Measure of performance: the regret on the quadratic error

$$R_n(\mathcal{A}) = \max_i L_n(\mathcal{A}) - \frac{\sum_{i=1}^N \sigma_i^2}{n}$$

Algorithm idea: mimic the behavior of the optimal strategy

$$T_{i,n} = \frac{\sigma_i^2}{\sum_{j=1}^N \sigma_j^2} n = \lambda_i n$$

The Active Bandit Problem

An UCB–based strategy

At each time step $t = 1, \ldots, n$

- Estimate

$$\hat{\sigma}^2_{i,T_{i,t-1}} = \frac{1}{T_{i,t-1}} \sum_{s=1}^{T_{i,t-1}} X^2_{s,i} - \hat{\mu}^2_{i,T_{i,t-1}}$$

- Compute

$$B_{i,t} = \frac{1}{T_{i,t-1}} \left(\hat{\sigma}^2_{i,T_{i,t-1}} + 5\sqrt{\frac{\log 1/\delta}{2\,T_{i,t-1}}} \right)$$

- Pull arm

$$I_t = \arg\max B_{i,t}$$

The Active Bandit Problem

Theorem

The UCB–based algorithm achieves a regret

$$R_n(\mathcal{A}) \leq \frac{98 \log(n)}{n^{3/2} \lambda_{\min}^{5/2}} + O\left(\frac{\log n}{n^2}\right)$$

Bibliography I

Reinforcement Learning

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