Approximate Dynamic Programming

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Value Iteration: the Idea

1. Let V_0 be *any* vector in R^N
2. At each iteration $k = 1, 2, \ldots, K$
   - Compute $V_{k+1} = \mathcal{T} V_k$
3. Return the *greedy* policy

$$\pi_K(x) \in \arg\max_{a \in A} \Big[r(x,a) + \gamma \sum_y p(y|x,a) V_K(y) \Big].$$

Value Iteration: the Guarantees

- From the *fixed point* property of $\mathcal{T}$:

$$\lim_{k \to \infty} V_k = V^*$$

- From the *contraction* property of $\mathcal{T}$

$$||V_{k+1} - V^*||_\infty \leq \gamma^{k+1} ||V_0 - V^*||_\infty \to 0$$

Problem: what if $V_{k+1} \neq \mathcal{T} V_k$??

Policy Iteration: the Idea

1. Let π_0 be *any* stationary policy
2. At each iteration $k = 1, 2, \ldots, K$
   - *Policy evaluation* given π_k, compute $V_k = V^{\pi_k}$.
   - *Policy improvement*: compute the *greedy* policy

$$\pi_{k+1}(x) \in \arg\max_{a \in A}\Big[r(x,a) + \gamma \sum_y p(y|x,a) V^{\pi_k}(y)\Big].$$

3. Return the last policy π_K

Policy Iteration: the Guarantees

The policy iteration algorithm generates a sequences of policies with *non-decreasing* performance

$$V^{\pi_{k+1}} \geq V^{\pi_k},$$

and it converges to π^* in a *finite* number of iterations.

Problem: what if $V_k \neq V^{\pi_k}$??

Sources of Error

- **Approximation error**. If X is *large* or *continuous*, value functions V cannot be *represented* correctly
  $\Rightarrow$ use an *approximation space* $\mathcal{F}$
- **Estimation error**. If the reward r and dynamics p are *unknown*, the Bellman operators $\mathcal{T}$ and $\mathcal{T}^\pi$ cannot be *computed* exactly
  $\Rightarrow$ *estimate* the Bellman operators from *samples*

In This Lecture

- Infinite horizon setting with discount γ
- Study the impact of approximation error
- Study the impact of estimation error in the next lecture

Outline

From Approximation Error to Performance Loss

Question: if V is an approximation of the optimal value function V^* with an error

$$\text{error} = \|V - V^*\|$$

how does it translate to the (loss of) performance of the *greedy policy*

$$\pi(x) \in \arg\max_{a \in A} \sum_y p(y|x,a)\big[r(x,a,y) + \gamma V(y)\big]$$

i.e.

$$\text{performance loss} = \|V^* - V^\pi\|$$

???

From Approximation Error to Performance Loss

Proposition

Let $V \in \mathbb{R}^N$ be an approximation of V^* and π its corresponding greedy policy, then

$$\underbrace{\|V^* - V^\pi\|_\infty}_{\textit{performance loss}} \leq \frac{2\gamma}{1-\gamma} \underbrace{\|V^* - V\|_\infty}_{\textit{approx. error}}.$$

Furthermore, there exists $\epsilon > 0$ such that if $\|V - V^*\|_\infty \leq \epsilon$, then π is *optimal*.

From Approximation Error to Performance Loss

Proof.

$$
\begin{aligned}
\|V^* - V^\pi\|_\infty &\leq \|\mathcal{T}V^* - \mathcal{T}^\pi V\|_\infty + \|\mathcal{T}^\pi V - \mathcal{T}^\pi V^\pi\|_\infty \\
&\leq \|\mathcal{T}V^* - \mathcal{T}V\|_\infty + \gamma\|V - V^\pi\|_\infty \\
&\leq \gamma\|V^* - V\|_\infty + \gamma(\|V - V^*\|_\infty + \|V^* - V^\pi\|_\infty) \\
&\leq \frac{2\gamma}{1-\gamma}\|V^* - V\|_\infty.
\end{aligned}
$$

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From Approximation Error to Performance Loss

Question: how do we compute V?

Problem: unlike in standard approximation scenarios (see supervised learning), we have a *limited access* to the target function, i.e. V^*

Objective: given an *approximation space $\mathcal{F}$*, compute an approximation V which is as close as possible to the *best approximation* of V^* in $\mathcal{F}$, i.e.

$$V \approx \arg \inf_{f \in \mathcal{F}} ||V^* - f||$$

Outline

Performance Loss

Approximate Value Iteration

Approximate Policy Iteration

Approximate Value Iteration: the Idea

Let $\mathcal{A}$ be an *approximation operator*.

1. Let V_0 be *any* vector in R^N
2. At each iteration $k = 1, 2, \ldots, K$
   - Compute $V_{k+1} = \mathcal{A}\mathcal{T} V_k$
3. Return the *greedy* policy

$$\pi_K(x) \in \arg\max_{a \in A} \Big[r(x,a) + \gamma \sum_y p(y|x,a) V_K(y) \Big].$$

Approximate Value Iteration: the Idea

Let $\mathcal{A} = \Pi_\infty$ be a projection operator in L_∞-norm, which corresponds to

$$V_{k+1} = \Pi_\infty \mathcal{T} V_k = \arg \inf_{V \in \mathcal{F}} \|\mathcal{T} V_k - V\|_\infty$$

Approximate Value Iteration: convergence

Proposition

The projection Π_∞ is a *non-expansion* and the joint operator $\Pi_\infty \mathcal{T}$ is a *contraction*.

Then there exists a unique fixed point $\tilde{V} = \Pi_\infty \mathcal{T} \tilde{V}$ which guarantees the *convergence* of AVI.

Approximate Value Iteration: performance loss

Proposition (Bertsekas & Tsitsiklis, 1996)

Let V^K be the function returned by AVI after K iterations and π_K its corresponding greedy policy. Then

$$\|V^* - V^{\pi_K}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \underbrace{\max_{0\leq k<K} \|\mathcal{T} V_k - \mathcal{A}\mathcal{T} V_k\|_\infty}_{\textit{worst approx. error}} + \frac{2\gamma^{K+1}}{1-\gamma} \underbrace{\|V^* - V_0\|_\infty}_{\textit{initial error}}.$$

Approximate Value Iteration: performance loss

Proof. Let $\varepsilon = \max_{0 \le k < K} \|\mathcal{T} V_k - \mathcal{A}\mathcal{T} V_k\|_\infty$.
For any $0 \le k < K$ we have

$$\begin{aligned} \|V^* - V_{k+1}\|_\infty &\le \|\mathcal{T} V^* - \mathcal{T} V_k\|_\infty + \|\mathcal{T} V_k - V_{k+1}\|_\infty \\ &\le \gamma \|V^* - V_k\|_\infty + \varepsilon, \end{aligned}$$

then

$$\begin{aligned} \|V^* - V_K\|_\infty &\le (1 + \gamma + \cdots + \gamma^{K-1})\varepsilon + \gamma^K \|V^* - V_0\|_\infty \\ &\le \frac{1}{1-\gamma}\varepsilon + \gamma^K \|V^* - V_0\|_\infty \end{aligned}$$

Since from Proposition 1 we have that
$\|V^* - V^{\pi_K}\|_\infty \le \frac{2\gamma}{1-\gamma}\|V^* - V_K\|_\infty$, then we obtain

$$\|V^* - V^{\pi_K}\|_\infty \le \frac{2\gamma}{(1-\gamma)^2}\varepsilon + \frac{2\gamma^{K+1}}{1-\gamma}\|V^* - V_0\|_\infty.$$

Fitted Q-iteration with linear approximation

Assumption: access to a generative model.

$$\begin{matrix} \text{State } x \\ \text{Action } a \end{matrix} \longrightarrow \boxed{\text{Generative model}} \longrightarrow \begin{matrix} \text{Reward} & r(x,a) \\ \text{Next state} & y \sim p(\cdot|x,a) \end{matrix}$$

Idea: work with Q-functions and linear spaces.

- Q^* is the unique fixed point of $\mathcal{T}$ defined over $X \times A$ as:
$$\mathcal{T} Q(x,a) = \sum_y p(y|x,a)[r(x,a,y) + \gamma \max_b Q(y,b)].$$
- $\mathcal{F}$ is a space defined by d features $\phi_1, \dots, \phi_d : X \times A \to \mathbb{R}$ as:
$$\mathcal{F} = \Big\{ Q_\alpha(x,a) = \sum_{j=1}^{d} \alpha_j \phi_j(x,a), \alpha \in \mathbb{R}^d \Big\}.$$

$\Rightarrow$ At each iteration compute $Q_{k+1} = \Pi_\infty \mathcal{T} Q_k$

Fitted Q-iteration with linear approximation

$\Rightarrow$ At each iteration compute $Q_{k+1} = \Pi_\infty \mathcal{T} Q_k$

Problems:

- the Π_∞ operator cannot be computed *efficiently*
- the Bellman operator $\mathcal{T}$ is often *unknown*

Fitted Q-iteration with linear approximation

Problem: the Π_∞ operator cannot be computed *efficiently*.

Let μ a distribution over X. We use a projection in $L_{2,\mu}$-norm onto the space $\mathcal{F}$:

$$Q_{k+1} = \arg\min_{Q \in \mathcal{F}} \|Q - \mathcal{T} Q_k\|_\mu^2.$$

Fitted Q-iteration with linear approximation

Problem: the Bellman operator $\mathcal{T}$ is often *unknown*.

1. Sample n state actions (X_i, A_i) with $X_i \sim \mu$ and A_i random,
2. Simulate $Y_i \sim p(\cdot|X_i, A_i)$ and $R_i = r(X_i, A_i, Y_i)$ with the generative model,
3. Estimate $\mathcal{T} Q_k(X_i, A_i)$ with

$$Z_i = R_i + \gamma \max_{a \in A} Q_k(Y_i, a)$$

(unbiased $\mathbb{E}[Z_i | X_i, A_i] = \mathcal{T} Q_k(X_i, A_i)$),

Fitted Q-iteration with linear approximation

At each iteration k compute Q_{k+1} as

$$Q_{k+1} = \arg\min_{Q_\alpha \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \left[Q_\alpha(X_i, A_i) - Z_i \right]^2$$

$\Rightarrow$ Since Q_α is a linear function in α, the problem is a simple *quadratic minimization* problem with *closed form* solution.

Other implementations

- K-nearest neighbour
- Regularized linear regression with L_1 or L_2 regularisation
- Neural network
- Support vector machine

Example: the Optimal Replacement Problem

State: level of wear of an object (e.g., a car).
Action: $\{(R)\text{eplace}, (K)\text{eep}\}$.
Cost:

- $c(x, R) = C$
- $c(x, K) = c(x)$ maintenance plus extra costs.

Dynamics:

- $p(\cdot|x, R) = \exp(\beta)$ with density $d(y) = \beta \exp^{-\beta y} \mathbb{I}\{y \geq 0\}$,
- $p(\cdot|x, K) = x + \exp(\beta)$ with density $d(y - x)$.

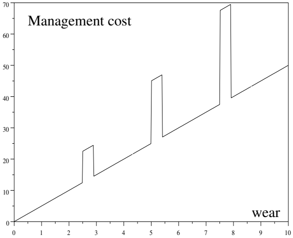
Problem: Minimize the discounted expected cost over an infinite horizon.

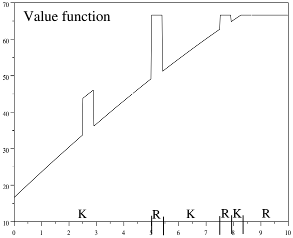
Example: the Optimal Replacement Problem

Optimal value function

$$V^*(x) = \min\left\{ c(x) + \gamma \int_0^\infty d(y-x)V^*(y)dy,\ C + \gamma \int_0^\infty d(y)V^*(y)dy \right\}$$

Optimal policy: action that attains the minimum





Linear approximation space $\mathcal{F} := \left\{ V_n(x) = \sum_{k=1}^{20} \alpha_k \cos(k\pi \frac{x}{x_{\max}}) \right\}$.

Example: the Optimal Replacement Problem

Collect N sample on a uniform grid.

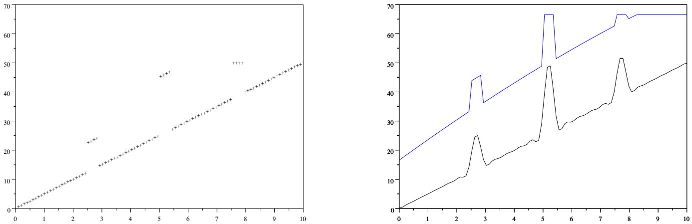


Figure: Left: the *target* values computed as $\{\mathcal{T}V_0(x_n)\}_{1\leq n\leq N}$. Right: the approximation $V_1 \in \mathcal{F}$ of the target function $\mathcal{T}V_0$.

Example: the Optimal Replacement Problem

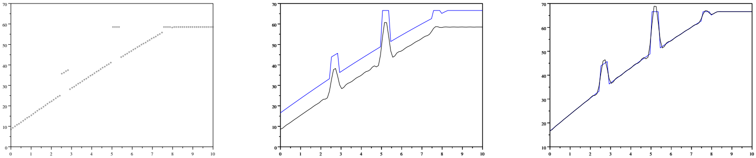


Figure: Left: the *target* values computed as $\{\mathcal{T} V_1(x_n)\}_{1 \leq n \leq N}$. Center: the approximation $V_2 \in \mathcal{F}$ of $\mathcal{T} V_1$. Right: the approximation $V_n \in \mathcal{F}$ after n iterations.

Outline

Performance Loss

Approximate Value Iteration

Approximate Policy Iteration

- Linear Temporal-Difference
- Least-Squares Temporal Difference
- Bellman Residual Minimization

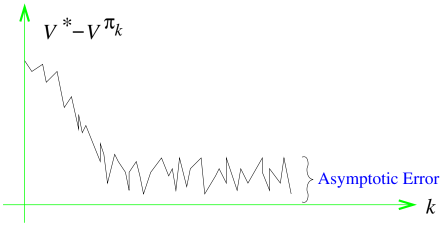
Approximate Policy Iteration: the Idea

Let $\mathcal{A}$ be an *approximation operator*.

- *Policy evaluation:* given the current policy π_k, compute $V_k = \mathcal{A}V^{\pi_k}$
- *Policy improvement:* given the *approximated* value of the current policy, compute the greedy policy w.r.t. V_k as

$$\pi_{k+1}(x) \in \arg\max_{a \in A} \Big[r(x,a) + \gamma \sum_{y \in X} p(y|x,a) V_k(y) \Big].$$

Problem: the algorithm is no longer guaranteed to converge.



Approximate Policy Iteration: performance loss

Proposition

The asymptotic performance of the policies π_k generated by the API algorithm is related to the approximation error as:

$$\limsup_{k\to\infty} \underbrace{\|V^* - V^{\pi_k}\|_\infty}_{\textit{performance loss}} \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k\to\infty} \underbrace{\|V_k - V^{\pi_k}\|_\infty}_{\textit{approximation error}}$$

Approximate Policy Iteration: performance loss

Proof. We introduce

- *Approximation error*: $e_k = V_k - V^{\pi_k}$,
- *Performance gain*: $g_k = V^{\pi_{k+1}} - V^{\pi_k}$,
- *Performance loss*: $l_k = V^* - V^{\pi_k}$.

Approximate Policy Iteration: performance loss

Proof (cont'd).
Since π_{k+1} is greedy w.r.t. V_k we have that $T^{\pi_{k+1}} V_k \geq T^{\pi_k} V_k$.

$$
\begin{aligned}
g_k &= T^{\pi_{k+1}} V^{\pi_{k+1}} - T^{\pi_{k+1}} V^{\pi_k} + T^{\pi_{k+1}} V^{\pi_k} - T^{\pi_{k+1}} V_k + T^{\pi_{k+1}} V_k - T^{\pi_k} V_k + T^{\pi_k} V_k - T^{\pi_k} V^{\pi} \\
&\overset{(a)}{\geq} \gamma P^{\pi_{k+1}} g_k - \gamma (P^{\pi_{k+1}} - P^{\pi_k})\, e_k \\
&\overset{(b)}{\geq} -\gamma (I - \gamma P^{\pi_{k+1}})^{-1} (P^{\pi_{k+1}} - P^{\pi_k})\, e_k
\end{aligned}
$$

Which leads to

$$
g_k \geq -\gamma (I - \gamma P^{\pi_{k+1}})^{-1} (P^{\pi_{k+1}} - P^{\pi_k})\, e_k, \tag{1}
$$

Approximate Policy Iteration: performance loss

Proof (cont'd).
Relationship between the performance at subsequent iterations. Since $T^{\pi^*} V_k \leq T^{\pi_{k+1}} V_k$ we have

$$\begin{aligned} l_{k+1} &= T^{\pi^*} V^* - T^{\pi^*} V^{\pi_k} + T^{\pi^*} V^{\pi_k} - T^{\pi^*} V_k \\ &\quad + T^{\pi^*} V_k - T^{\pi_{k+1}} V_k + T^{\pi_{k+1}} V_k - T^{\pi_{k+1}} V^{\pi_k} \\ &\quad + T^{\pi_{k+1}} V^{\pi_k} - T^{\pi_{k+1}} V^{\pi_{k+1}} \\ &\leq \gamma[P^{\pi^*} l_k - P^{\pi_{k+1}} g_k + (P^{\pi_{k+1}} - P^{\pi^*}) e_k]. \end{aligned}$$

If we now plug-in equation (1),

$$\begin{aligned} l_{k+1} &\leq \gamma P^{\pi^*} l_k + \gamma[P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1}(P^{\pi_{k+1}} - P^{\pi_k}) + P^{\pi_{k+1}} - P^{\pi^*}] e_k \\ &\leq \gamma P^{\pi^*} l_k + \gamma[P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1}(I - \gamma P^{\pi_k}) - P^{\pi^*}] e_k. \end{aligned}$$

Thus we obtain the fact that the performance loss changes through iterations as

$$l_{k+1} \quad \leq \quad \gamma P^{\pi^*} l_k + \gamma[P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1}(I - \gamma P^{\pi_k}) - P^{\pi^*}] e_k.$$

Approximate Policy Iteration: performance loss

Proof (cont'd).
Move to asymptotic regime.
Let $f_k = \gamma[P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1}(I - \gamma P^{\pi_k}) - P^{\pi^*}]e_k$, we have

$$l_{k+1} \leq \gamma P^{\pi^*} l_k + f_k,$$

thus if we move to the $\limsup$ we obtain,

$$\begin{aligned} (I - \gamma P^{\pi^*}) \limsup_{k\to\infty} l_k &\leq \limsup_{k\to\infty} f_k \\ \limsup_{k\to\infty} l_k &\leq (I - \gamma P^{\pi^*})^{-1} \limsup_{k\to\infty} f_k, \end{aligned}$$

since $I - \gamma P^{\pi^*}$ is invertible. Finally, we only need to take the L_∞-norm both sides and obtain,

$$\begin{aligned} \limsup_{k\to\infty} \|l_k\| &\leq \frac{\gamma}{1-\gamma} \limsup_{k\to\infty} \|P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1}(I + \gamma P^{\pi_k}) + P^{\pi^*}\| \, \|e_k\| \\ &\leq \frac{\gamma}{1-\gamma}\Big(\frac{1+\gamma}{1-\gamma} + 1\Big) \limsup_{k\to\infty} \|e_k\| = \frac{2\gamma}{(1-\gamma)^2} \limsup_{k\to\infty} \|e_k\|. \end{aligned}$$

Outline

Linear TD(λ): the algorithm

Algorithm Definition

Given a *linear space* $\mathcal{F} = \{V_\alpha(x) = \sum_{i=1}^d \alpha_i \phi_i(x), \alpha \in \mathbb{R}^d\}$.
Trace vector $z \in \mathbb{R}^d$ and *parameter* vector $\alpha \in \mathbb{R}^d$ initialized to zero.
Generate a sequence of states $(x_0, x_1, x_2, \dots)$ according to π.
At each step t, the *temporal difference* is

$$d_t = r(x_t, \pi(x_t)) + \gamma V_{\alpha_t}(x_{t+1}) - V_{\alpha_t}(x_t)$$

and the parameters are updated as

$$\begin{aligned} \alpha_{t+1} &= \alpha_t + \eta_t d_t z_t, \\ z_{t+1} &= \lambda \gamma z_t + \phi(x_{t+1}), \end{aligned}$$

where η_t is learning step.

Linear TD(λ): approximation error

Proposition (Tsitsiklis et Van Roy, 1996)

Let the learning rate η_t satisfy

$$\sum_{t\geq 0} \eta_t = \infty, \text{ and } \sum_{t\geq 0} \eta_t^2 < \infty.$$

We assume that π admits a *stationary distribution* μ_π and that the features $(\phi_i)_{1\leq k\leq K}$ are *linearly independent*. There exists a fixed α^* such that

$$\lim_{t\to\infty} \alpha_t = \alpha^*.$$

Furthermore we obtain

$$\underbrace{\|V_{\alpha^*} - V^\pi\|_{2,\mu^\pi}}_{\textit{approximation error}} \leq \frac{1-\lambda\gamma}{1-\gamma} \underbrace{\inf_\alpha \|V_\alpha - V^\pi\|_{2,\mu^\pi}}_{\textit{smallest approximation error}}.$$

Linear TD(λ): approximation error

Remark: for $\lambda = 1$, we recover Monte-Carlo (or TD(1)) and the bound is the smallest!

Problem: the bound does not consider the variance (i.e., samples needed for α_t to converge to α^*).

Linear TD(λ): implementation

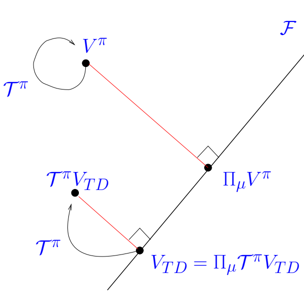
- **Pros**: simple to implement, computational cost *linear* in d.
- **Cons**: very sample *inefficient*, many samples are needed to converge.

Outline

Least-squares TD: the algorithm

Recall: $V^\pi = \mathcal{T}^\pi V^\pi$.

Intuition: compute $V = \mathcal{A}\mathcal{T}^\pi V$.



Focus on the $L_{2,\mu}$-weighted norm and projection Π_μ

$$\Pi_\mu g = \arg\min_{f\in\mathcal{F}} \|f - g\|_\mu.$$

Least-squares TD: the algorithm

By construction, the Bellman residual of V_{TD} is orthogonal to $\mathcal{F}$, thus for any $1 \leq i \leq d$

$$\langle \mathcal{T}^\pi V_{TD} - V_{TD}, \phi_i \rangle_\mu = 0,$$

and

$$\begin{aligned} \langle r^\pi + \gamma P^\pi V_{TD} - V_{TD}, \phi_i \rangle_\mu &= 0 \\ \langle r^\pi, \phi_i \rangle_\mu + \sum_{j=1}^{d} \langle \gamma P^\pi \phi_j - \phi_j, \phi_i \rangle_\mu \alpha_{TD,j} &= 0, \end{aligned}$$

$\Rightarrow$ α_{TD} is the solution of a *linear system* of order d.

Least-squares TD: the algorithm

Algorithm Definition

The LSTD solution α_{TD} can be computed by computing the matrix A and vector b defined as

$$\begin{array}{rcl} A_{i,j} &=& \langle \phi_i, \phi_j - \gamma P^\pi \phi_j \rangle_\mu \\ b_i &=& \langle \phi_i, r^\pi \rangle_\mu \end{array},$$

and then solving the system $A\alpha = b$.

Least-squares TD: the approximation error

Problem: in general $\Pi_\mu \mathcal{T}^\pi$ does not admit a fixed point (i.e., matrix A is not invertible).

Solution: use the stationary distribution μ_π of policy π, that is

$$\mu_\pi P^\pi = \mu_\pi, \text{ and } \mu_\pi(y) = \sum_x p(y|x, \pi(x))\mu_\pi(x)$$

Least-squares TD: the approximation error

Proposition

The Bellman operator $\mathcal{T}^\pi$ is a *contraction* in the weighted L_{2,μ_π}-norm. Thus the joint operator $\Pi_{\mu_\pi}\mathcal{T}^\pi$ is a contraction and it admits a unique *fixed point* V_{TD}. Then

$$\underbrace{||V^\pi - V_{TD}||_{\mu_\pi}}_{\textit{approximation error}} \leq \frac{1}{\sqrt{1-\gamma^2}} \underbrace{\inf_{V\in\mathcal{F}} ||V^\pi - V||_{\mu_\pi}}_{\textit{smallest approximation error}} .$$

Least-squares TD: the approximation error

Proof.
We show that $\|P_\pi\|_{\mu_\pi} = 1$:

$$\begin{aligned} \|P^\pi V\|^2_{\mu_\pi} &= \sum_x \mu_\pi(x)\Big(\sum_y p(y|x,\pi(x))V(y)\Big)^2 \\ &\leq \sum_x \sum_y \mu_\pi(x) p(y|x,\pi(x))V(y)^2 \\ &= \sum_y \mu_\pi(y)V(y)^2 = \|V\|^2_{\mu_\pi}. \end{aligned}$$

It follows that $\mathcal{T}^\pi$ is a contraction in L_{2,μ_π}, i.e.,

$$\|\mathcal{T}^\pi V_1 - \mathcal{T}^\pi V_2\|_{\mu_\pi} = \gamma\|P^\pi(V_1 - V_2)\|_{\mu_\pi} \leq \gamma\|V_1 - V_2\|_{\mu_\pi}.$$

Thus $\Pi_{\mu_\pi}\mathcal{T}^\pi$ is a composition of a non-expansion and a contraction in L_{2,μ_π}, thus $V_{TD} = \Pi_{\mu_\pi}\mathcal{T}^\pi V_{TD}$.

Least-squares TD: the approximation error

Proof.
By Pythagorean theorem we have

$$||V^\pi - V_{TD}||^2_{\mu_\pi} = ||V^\pi - \Pi_{\mu_\pi} V^\pi||^2_{\mu_\pi} + ||\Pi_{\mu_\pi} V^\pi - V_{TD}||^2_{\mu_\pi},$$

but

$$||\Pi_{\mu_\pi} V^\pi - V_{TD}||^2_{\mu_\pi} = ||\Pi_{\mu_\pi} V^\pi - \Pi_{\mu_\pi} \mathcal{T}^\pi V_{TD}||^2_{\mu_\pi} \leq ||\mathcal{T}^\pi V^\pi - \mathcal{T} V_{TD}||^2_{\mu_\pi} \leq \gamma^2 ||V^\pi$$

Thus

$$||V^\pi - V_{TD}||^2_{\mu_\pi} \leq ||V^\pi - \Pi_{\mu_\pi} V^\pi||^2_{\mu_\pi} + \gamma^2 ||V^\pi - V_{TD}||^2_{\mu_\pi},$$

which corresponds to eq.(**??**) after reordering.

Least-squares TD: the implementation

- Generate $(X_0, X_1, \dots)$ from *direct execution* of π and observes $R_t = r(X_t, \pi(X_t))$
- Compute estimates

$$\hat{A}_{ij} = \frac{1}{n}\sum_{t=1}^{n} \phi_i(X_t)[\phi_j(X_t) - \gamma\phi_j(X_{t+1})],$$

$$\hat{b}_i = \frac{1}{n}\sum_{t=1}^{n} \phi_i(X_t)R_t.$$

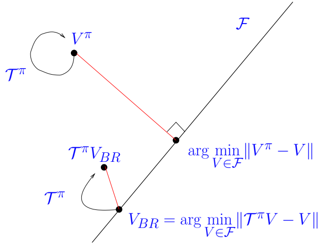
- Solve $\hat{A}\alpha = \hat{b}$

Remark:

- No need for a generative model.
- If the chain is ergodic, $\hat{A} \to A$ et $\hat{b} \to b$ when $n \to \infty$.

Outline

Bellman Residual Minimization (BRM): the idea



Let μ be a distribution over X, V_{BR} is the minimum *Bellman residual w.r.t.* $\mathcal{T}^\pi$

$$V_{BR} = \arg\min_{V\in\mathcal{F}} \| T^\pi V - V \|_{2,\mu}$$

Bellman Residual Minimization (BRM): the idea

The mapping $\alpha \to \mathcal{T}^\pi V_\alpha - V_\alpha$ is affine
The function $\alpha \to \|\mathcal{T}^\pi V_\alpha - V_\alpha\|_\mu^2$ is quadratic
$\Rightarrow$ The minimum is obtained by computing the *gradient and setting it to zero*

$$\langle r^\pi + (\gamma P^\pi - I) \sum_{j=1}^{d} \phi_j \alpha_j, (\gamma P^\pi - I)\phi_i \rangle_\mu = 0,$$

which can be rewritten as $A\alpha = b$, with

$$\begin{cases} A_{i,j} = \langle \phi_i - \gamma P^\pi \phi_i, \phi_j - \gamma P^\pi \phi_j \rangle_\mu, \\ b_i = \langle \phi_i - \gamma P^\pi \phi_i, r^\pi \rangle_\mu, \end{cases}$$

Bellman Residual Minimization (BRM): the idea

Remark: the system admits a solution whenever the features ϕ_i are *linearly independent* w.r.t. μ

Remark: let $\{\psi_i = \phi_i - \gamma P^\pi \phi_i\}_{i=1\ldots d}$, then the previous system can be interpreted as a linear regression problem

$$\|\alpha \cdot \psi - r^\pi\|_\mu$$

BRM: the approximation error

Proposition

We have

$$\|V^\pi - V_{BR}\| \leq \|(I - \gamma P^\pi)^{-1}\|(1 + \gamma\|P^\pi\|) \inf_{V \in \mathcal{F}} \|V^\pi - V\|.$$

If μ_π is the *stationary policy* of π, then $\|P^\pi\|_{\mu_\pi} = 1$ and $\|(I - \gamma P^\pi)^{-1}\|_{\mu_\pi} = \frac{1}{1-\gamma}$, thus

$$\|V^\pi - V_{BR}\|_{\mu_\pi} \leq \frac{1+\gamma}{1-\gamma} \inf_{V \in \mathcal{F}} \|V^\pi - V\|_{\mu_\pi}.$$

BRM: the approximation error

Proof. We relate the Bellman residual to the approximation error as

$$\begin{aligned} V^\pi - V &= V^\pi - T^\pi V + T^\pi V - V = \gamma P^\pi (V^\pi - V) + T^\pi V - \\ (I - \gamma P^\pi)(V^\pi - V) &= T^\pi V - V, \end{aligned}$$

taking the norm both sides we obtain

$$\|V^\pi - V_{BR}\| \leq \|(I - \gamma P^\pi)^{-1}\| \|\mathcal{T}^\pi V_{BR} - V_{BR}\|$$

and

$$\|\mathcal{T}^\pi V_{BR} - V_{BR}\| = \inf_{V \in \mathcal{F}} \|\mathcal{T}^\pi V - V\| \leq (1 + \gamma \|P^\pi\|) \inf_{V \in \mathcal{F}} \|V^\pi - V\|.$$

BRM: the approximation error

Proof. If we consider the stationary distribution μ_π, then $\|P^\pi\|_{\mu_\pi} = 1$. The matrix $(I - \gamma P^\pi)$ can be written as the power series $\sum_t \gamma (P^\pi)^t$. Applying the norm we obtain

$$\|(I - \gamma P^\pi)^{-1}\|_{\mu_\pi} \leq \sum_{t \geq 0} \gamma^t \|P^\pi\|_{\mu_\pi}^t \leq \frac{1}{1-\gamma}$$

■

BRM: the implementation

Assumption. A generative model is available.

- Drawn n states $X_t \sim \mu$
- Call generative model on (X_t, A_t) (with $A_t = \pi(X_t)$) and obtain $R_t = r(X_t, A_t)$, $Y_t \sim p(\cdot|X_t, A_t)$
- Compute

$$\hat{\mathcal{B}}(V) = \frac{1}{n}\sum_{t=1}^{n}\big[V(X_t) - \underbrace{\big(R_t + \gamma V(Y_t)\big)}_{\hat{\mathcal{T}}V(X_t)}\big]^2.$$

BRM: the implementation

Problem: this estimator is *biased and not consistent*! In fact,

$$
\begin{aligned}
\mathbb{E}[\hat{\mathcal{B}}(V)] &= \mathbb{E}\Big[\big[V(X_t) - \mathcal{T}^\pi V(X_t) + \mathcal{T}^\pi V(X_t) - \hat{\mathcal{T}} V(X_t)\big]^2\Big] \\
&= \|\mathcal{T}^\pi V - V\|_\mu^2 + \mathbb{E}\Big[\big[\mathcal{T}^\pi V(X_t) - \hat{\mathcal{T}} V(X_t)\big]^2\Big]
\end{aligned}
$$

$\Rightarrow$ minimizing $\hat{\mathcal{B}}(V)$ *does not* correspond to minimizing $\mathcal{B}(V)$ (even when $n \to \infty$).

BRM: the implementation

Solution. In each state X_t, generate *two independent samples* Y_t et $Y'_t \sim p(\cdot|X_t, A_t)$
Define

$$\hat{\mathcal{B}}(V) = \frac{1}{n}\sum_{t=1}^{n}\big[V(X_t) - \big(R_t + \gamma V(Y_t)\big)\big]\big[V(X_t) - \big(R_t + \gamma V(Y'_t)\big)\big].$$

$\Rightarrow \hat{\mathcal{B}} \to \mathcal{B}$ for $n \to \infty$.

BRM: the implementation

The function $\alpha \to \hat{\mathcal{B}}(V_\alpha)$ is quadratic and we obtain the linear system

$$\begin{aligned}
\widehat{A}_{i,j} &= \frac{1}{n}\sum_{t=1}^{n}\big[\phi_i(X_t) - \gamma\phi_i(Y_t)\big]\big[\phi_j(X_t) - \gamma\phi_j(Y'_t)\big],\\
\widehat{b}_i &= \frac{1}{n}\sum_{t=1}^{n}\big[\phi_i(X_t) - \gamma\frac{\phi_i(Y_t) + \phi_i(Y'_t)}{2}\big] R_t.
\end{aligned}$$

LSTD vs BRM

- **Different assumptions:** BRM requires a *generative model*, LSTD requires a *single trajectory*.
- **The performance is evaluated differently:** BRM *any* distribution, LSTD *stationary* distribution μ^π.

Bibliography I

Reinforcement Learning

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