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Algebraic differentiators through orthogonal polynomials series expansions

Rosane Ushirobira

Inria, Non-A Team, Villeneuve d'Ascq, France

ABSTRACT
Numerical differentiation is undoubtedly a fundamental problem in signal processing and control engineering, due to its countless applications. The goal of this paper is to address this question within an algebraic framework. More precisely, we consider a noisy signal and its orthogonal polynomial series expansion. Through the algebraic identification of the series coefficients, we then propose algebraic differentiators for the signal. Examples based on Hermite and Laguerre polynomials illustrate these algebraic differentiators.

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1. Introduction

Numerical differentiation is certainly a fundamental topic in signal processing and control engineering. It emerges in numerous problems concerning the reconstruction of a noisy signal, parameter estimation, state observation, and so on, this list being extensively long. In this way, numerical differentiation has been the subject of countless research works.

To solve this sort of estimation problems, most classical methods involve usually statistical approaches, for instance, the minimum least squares method (Duncan, Mandl, & Psik-Duncan, 1996) or the maximum likelihood for parameter estimation in sinusoidal signals (Edmonson, Lee, & Anderson, 1995). Nevertheless, ingenious new deterministic methods have appeared in last decades providing robustness to noise results. In this context, we may cite Levant (1998, 2003) where derivatives were estimated using sliding-mode tools and (Efimov & Fridman, 2011) where a higher order sliding-mode-based differentiator is proposed. In Perruquetti, Floquet, and Moulay (2008), homogeneous-based tools were used to design finite-time observers. The reader may refer to the many subsequent papers using these three works. These differentiators were also applied in practical domains in different contexts. For instance, algebraic and homogeneous-based numerical differentiations were used in Ahmed, Ushirobira, Efimov, Tran, and Massabuau (2015) to estimate the velocity of valve movement in marine bivalve mollusks for water quality surveillance. They were also applied to a fault detection-like problem in Ahmed et al. (2016) to detect the spawning time of the bivalve mollusks. Very recently, Ushirobira, Efimov, Casiez, Roussel, and Perruquetti (2016) used these deterministic differentiators in a forecasting algorithm for the latency compensation in indirect human–computer interactions.

In the early 2000s, an algebraic framework began to be widespread, notably in the study of parameter identification. An important work in this subject is Fliess and Sira-Ramírez (2003) where the authors introduce a closed-loop parametric identification procedure for continuous-time constant linear systems. Algebraic strategies are generally based on differential algebra concepts, operational calculus and module theory. The book (Sira-Ramírez, Rodríguez, Romero, & Juárez, 2014) contains a recent survey on algebraic identification. Several applications in various different problems using the algebraic approach can be found in Mboup (2012), Mboup (2009), Perruquetti, Bonnet, Mboup, Ushirobira, and Fraisse (2012), Cortés-Romero, García-Rodríguez, Luviano-Juárez, and Sira-Ramírez (2011), Menhour, d'Andrea Novel, Boussard, Fliess, and Mounier (2011), Ushirobira, Perruquetti, Mboup, and Fliess (2011), and Ushirobira, Perruquetti, and Mboup (2016a).

Parallel to the numerical differentiation issue, there is the essential problem of the reconstruction of a signal from a noisy measurement. A common approach to this problem is to use a Taylor series expansion of the signal. In simple words, the idea is to approximate the signal by a truncation of its Taylor series expansion. It is mostly in this way that numerical differentiation has been the centre of attention in many papers. The algebraic approach in this context started with the noteworthy paper (Mboup, Join, & Fliess, 2009) which inspired the present work and many others(see, for instance, Liu, Gibaru, & Perruquetti, 2011a, 2011b; Mboup, 2009).

An alternative method for numerical differentiation is to consider the series expansion of the signal in a different basis. For example, in signal processing, a common signal decomposition is provided by a series expansion in an orthogonal polynomial basis (i.e. the signal is written as an infinite sum of orthogonal polynomials). The goal in this case is to identify the coefficients in this series expansion. In Ushirobira and Quadrat (2016), the algebraic framework within this context started to be studied and the results in the present paper are in part based on this former work. In Ushirobira and Quadrat (2016), the authors also studied dynamical systems described by some particular second-order ordinary differential equation (ODE) with orthogonal polynomials as solutions.

The algebraic method proposed in this paper is greatly based on the structural properties of the *Weyl algebra*. A major advantage in this sense is to obtain closed formulas for the differentiators. Similar approaches can be found in Ushirobira,

CONTACT Rosane Ushirobira rosane.ushirobira@inria.fr

Perruquetti, Mboup, and Fliess (2012) and Ushirobira, Perruquetti, Mboup, and Fliess (2013).

2. General problem

All through this paper:

- $\mathbb{K}$ denotes a field of characteristic zero (e.g. $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$);
- for a set $\mathcal{S}$, the algebraic extension of $\mathbb{K}$ by $\mathcal{S}$ is denoted by $\mathbb{K}(\mathcal{S})$.

2.1 Orthogonal polynomials expansion

In this section, the algebraic-based reconstruction of a signal via orthogonal polynomials is presented. Let us consider a signal x to be recovered from a noisy signal y defined by

$$y(t) = x(t) + \varpi(t),$$

where $\varpi(t)$ is a zero-mean noise.[1]

We denote by $\mathscr{G}$ the basis formed by generic classical orthogonal polynomials g_n, $\mathscr{G} = \left\{g_n(t), \langle \cdot, \cdot \rangle\right\}_{n\geq 0}$ where $\langle \cdot, \cdot \rangle$ denote the scalar product on the g_n with a corresponding weight function w (see the Appendix). The expansion of a continuous signal y in $\mathscr{G}$ can be described by

$$y(t) = \sum_{n\geq 0} \lambda_n \, g_n(t), \qquad (1)$$

where $\lambda_n \in \mathbb{R}$ corresponds to the projection of y onto the orthogonal basis $\mathscr{G}$:

$$\lambda_n = \frac{\langle y(t), g_n(t) \rangle}{\langle g_n(t), g_n(t) \rangle} = \frac{\int_{\mathbb{R}} y(t)\, g_n(t)\, w(t) dt}{\int_{\mathbb{R}} g_n(t)^2\, w(t) dt}. \qquad (2)$$

The expansion (1) is to be considered as a formal series. The convergence of (1) to $y(t)$ at a time t in a given interval I where the functions of $\mathscr{G}$ remain bounded for $n \to \infty$ depends uniquely upon the nature of the signal $y(t)$ in the neighbourhood of t. All classical orthogonal polynomials are uniformly bounded in the orthogonality interval. More details can be found in Shohat (1935).

Remark that some parameters might appear in the polynomials g_n, for example $\alpha, \beta > -1$ in the Jacobi polynomials $P_n^{(\alpha,\beta)}$ or $\alpha > -1$ in the Laguerre polynomials $\ell_n^{(\alpha)}$ (more details in Szegö, 1975).

To reconstruct the signal x, the whole family of parameters $\{\lambda_n\}_{n\in\mathbb{N}}$ would have to be estimated. However, for a good approximation, it might be enough to estimate only a finite number of these parameters. Moreover, notice that y will often represent the measured signal from a signal x with some negligible noise, so we may consider only y. Hence, based on the similar case of the Taylor expansion (see e.g. Mboup et al., 2009), an approximation of y will be given by a truncated series:

$$y(t) \approx y_N(t) = \sum_{n=0}^{N} \lambda_n \, g_n(t), \qquad (3)$$

for some $N > 0$. The idea is then to identify the coefficients $\lambda_0, \ldots, \lambda_N$. It is worth to notice that in Mboup et al. (2009), the authors establish the strong connection between numerical estimators proposed in that work and Jacobi orthogonal polynomials.

In this work, the algebraic estimation method is proposed via computations in the operational domain. In other words, a passage is required from the time domain to the Laplace domain via the Laplace transform $\mathscr{L}$. Let us recall the action of $\mathscr{L}$ on a continuous function f with support in $\mathbb{R}_+$ given by

$$\mathscr{L}\left(f\right)(s) = \int_0^{+\infty} e^{-st} \, f(t)\, dt,$$

where s denotes the Laplace variable. Then, $\mathscr{L}$ applied on (3) yields

$$Y_N(s) = \sum_{n=0}^{N} \lambda_n \, \mathscr{L}\left(g_n\right)(s), \qquad (4)$$

where Y_N denotes the Laplace transform of y_N. Our goal is then to estimate the constants λ_i, $i = 0, \ldots, N$.

Prior to summarising the estimation method, we recall the definition of the inverse Laplace transform that allows the return to the time domain from the Laplace domain:

$$\mathscr{L}^{-1}\left(\frac{1}{s^m}\frac{d^pY(s)}{ds^p}\right) = \frac{(-1)^p}{(m-1)!}\int_0^t \upsilon_{m-1,p}(\tau)\, y(\tau) d\tau \qquad (5)$$

with Y denoting the Laplace transform of y and

$$\upsilon_{m,p} = (t-\tau)^m \, \tau^p, \forall\, p, m \in \mathbb{N}, m \geq 1. \qquad (6)$$

The steps of the algebraic method proposed in this paper are schematised below:

- conversion of the truncated time-dependent orthogonal expansion to an algebraic finite sum in the Laplace domain;
- for each λ_i, gradual elimination of other parameters through the action of differential operators called *annihilators*;
- return to the time domain through the action of the inverse Laplace transform, providing closed formulas for λ_i.

Proposition 2.1 gives the well-known properties of the Laplace transform that will be useful in the sequel:

Proposition 2.1: *Let F be the Laplace transform of a continuous function f. Then,*

(1) $\mathscr{L}\left(\frac{d^n f}{dt^n}\right)(s) = s^n F(s) - \sum_{i=0}^{n-1} s^{n-i-1}\frac{d^i f}{dt^i}(0)$.
(2) $\mathscr{L}\left(t^n\, f\right)(s) = (-1)^n \frac{d^n}{ds^n}F(s)$.
(3) *For all* $a \in \mathbb{K}$, $\mathscr{L}\,(a) = \frac{a}{s}$.

2.2 Orthogonal differentiators

The numerical differentiation problem can be addressed in this orthogonal series expansion framework in the following manner. In the previous subsection, an algebraic estimation approach was sketched to estimate the parameters appearing in the expansion. To approximate derivatives by integrals, for given orthogonal polynomials g_n, we will start by defining constants h_n and k_n as in Diekema and Koornwinder (2012) by

$$h_n := \int_{\mathbb{R}} g_n(t)^2 w(t)dt \quad \text{and}$$
$$g_n(t) = k_n t^n + \text{ terms of degree less than } n$$

where w is the weight function corresponding to g_n. The following Lemma is straightforward (Diekema & Koornwinder, 2012):

Lemma 2.1: *We have* $\int_{\mathbb{R}} t^n\, g_n(t)\, w(t)dt = \frac{h_n}{k_n}$.

Examples 2.1: *For instance, these constants are*

(1) *in the case of Hermite polynomials:*

$$h_n = \sqrt{\pi}\, 2^n\, n!, \qquad k_n = 2^n.$$

(2) *in the case of Laguerre polynomials:*

$$h_n = \frac{\Gamma(n+\alpha+1)}{n!}, \qquad k_n = \frac{(-1)^n}{n!}.$$

The relation between the coefficients of the orthogonal polynomials series expansion that were named λ_n and estimate derivatives of the signal y are expressed in Theorem 2.1:

Theorem 2.1: *Consider a signal y as in the previous subsection and its expansion in (1), (3). Then, the nth-order time-derivative estimate $\widehat{y^{(n)}}$ of y is given by*

$$\widehat{y^{(n)}}(t) = n!\, k_n\, \lambda_n \tag{7}$$

Proof: The Taylor series expansion of y is given by the following expression:

$$y(t) = \sum_{i\geq 0} \frac{a_i}{i!} t^i$$

where the terms a_i are the unknown constant coefficients that represent the derivatives of the signal y. Assume that the derivative estimation can be done on a moving time window of length $T > 0$, so $a_i = y^{(i)}(t)$, for all $t > T$. Using a truncation of order n of this expansion (3), the orthogonal expansion in (3) and (2), we obtain

$$\begin{aligned}\langle y_n(t), g_n(t)\rangle &= \int_{\mathbb{R}} y_n(t)\, g_n(t)\, w(t)dt\\ &= \int_{\mathbb{R}} \sum_{i=0}^{n} \frac{a_i}{i!}\, t^i\, g_n(t)\, w(t)dt\end{aligned}$$

By orthogonality and from Lemma 2.1, that provides

$$\langle y_n(t), g_n(t)\rangle = \frac{a_n}{n!} \int_{\mathbb{R}} t^n\, g_n(t)\, w(t)dt = \frac{h_n}{k_n}\, \frac{a_n}{n!}$$

Using now the orthogonal series decomposition of y on the basis $\mathscr{G}$ and also (2), we obtain

$$a_n \approx n!\, \langle g_n(t), g_n(t)\rangle\, \frac{k_n}{h_n}\, \lambda_n \;= n!\, k_n\, \lambda_n$$

That finishes the proof. □

Therefore, Theorem 2.1 provides a useful formula for differentiators by linking the orthogonal coefficients projections and the derivative estimations.

3. Illustrating examples

3.1 Hermite polynomials

We start by considering a particular type of orthogonal polynomials, *Hermite polynomials* h_n. They form an orthogonal set for $t \in \mathbb{R}$ with respect to the weight function e^{-t^2} (see the Appendix). Given a continuous function y, its Hermite expansion can be written as

$$y(t) = \sum_{n\geq 0} \lambda_n\, h_n(t), \tag{8}$$

where

$$\lambda_n = \frac{1}{2^n\, n!\, \sqrt{\pi}} \int_{-\infty}^{\infty} y(\tau)\, h_n(\tau)\, \mathrm{e}^{-\tau^2}\, d\tau.$$

An approximation of the function y is provided by selecting a constant $N > 0$ and it follows from (4)

$$Y_N(s) = \sum_{n=0}^{N} \lambda_n\, H_n(s), \tag{9}$$

where Y_N denotes the Laplace transform of y_N and H_n the Laplace transform of h_n. From the definition of Hermite polynomials (see the Appendix), it follows

$$h_n(t) = 2^n t^n + he_n(t) = 2^n t^n + \eta_{n,n-2} t^{n-2} + \cdots + \eta_{n,m} t^m$$

with $m = n \bmod 2$ (i.e. $m = 0$ if n is even and $m = 1$ if n is odd). Writing $n = 2j$ or $n = 2j + 1$, it results that

$$H_n(s) := \mathscr{L}(h_n)(s) = 2^n \frac{n!}{s^{n+1}} + \sum_{k=1}^{j} \eta_{n,n-2k} \frac{(n-2k)!}{s^{n-2k+1}}.$$

Eliminating denominators in (9) provides

$$\begin{aligned} s^{N+1} Y_N(s) = \lambda_N \left(2^N N! + \sum_{k=1}^{j} \eta_{N,N-2k} (N-2k)!\, s^{2k} \right) \\ + \sum_{n=0}^{N-1} \lambda_n s^{N+1} H_n(s). \end{aligned} \tag{10}$$

In this work, we denote the set of parameters to be estimated by

$$\Theta := \{\lambda_0, \lambda_1, \ldots, \lambda_N\}.$$

Our goal here is the individual identification of these parameters, starting with the dominant coefficient λ_N. Let us adopt the notation $\Theta_{\text{est}} = \{\lambda_N\}$.

The expression obtained in (10) may be rewritten as a relation denoted by $(\mathscr{R})$:

$$(\mathscr{R}) \qquad P\,(s)\,Y_N(s) + Q(s) + \overline{Q}(s) = 0, \tag{11}$$

where $P \in \mathbb{R}_\Theta\left[s, \frac{d}{ds}\right]$ (i.e. P is a differential operator on the Laplace variable s with coefficients in the field $\mathbb{R}_\Theta := \mathbb{R}\,(\Theta)$), $Q \in \mathbb{R}_{\Theta_{\text{est}}}[s]$ (i.e. Q is a polynomial in s with coefficients in $\mathbb{R}_{\Theta_{\text{est}}} := \mathbb{R}\,(\Theta_{\text{est}}))$, and $\overline{Q} \in \mathbb{R}_\Theta\,[s]$ (i.e. $\overline{Q}$ is a polynomial in s with coefficients in $\mathbb{R}_\Theta$):

$$\begin{cases} P(s) = s^{N+1}, \\ Q(s) = -\lambda_N\, s^{N+1} H_N(s), \\ \overline{Q}(s) = -\displaystyle\sum_{n=0}^{N-1} \lambda_n\, s^{N+1} H_n(s). \end{cases}$$

To obtain an expression from (11) that involves only the parameter λ_N, the polynomial $\overline{Q}$ must somehow be eliminated. It is through the action of differential operators that this elimination is realised. These operators are called *annihilators* and their goal is to provide an expression containing only λ_N, Y_N and its derivatives. The definition of annihilators will be given in the next section. A time-domain closed formula is obtained by applying the inverse Laplace transform on the resulting equation.

When $N = 2$ for instance, it follows from (10) that

$$s^3\, Y_N\,(s) - s^2\,\lambda_0 - 2\,s\,\lambda_1 + 2\,\left(s^2 - 4\right)\,\lambda_2 = 0. \tag{12}$$

Set $\Theta_{\text{est}} = \{\lambda_2\}$ to obtain

$$\begin{aligned} &P(s) = s^3, \quad Q(s) = 2\,\left(s^2 - 4\right)\lambda_2, \\ &\overline{Q}(s) = -s^2\,\lambda_0 - 2\,s\,\lambda_1. \end{aligned} \tag{13}$$

3.2 Laguerre polynomials

(Generalised) Laguerre polynomials $\ell_n^{(\alpha)}$ are another type of classical orthogonal polynomials. The weight function $w(t) = t^{\alpha-1}\,\mathrm{e}^{-t}$ $(\alpha > -1)$ provides the orthogonality of the family $\{\ell_n^{(\alpha)}\}$ with respect to the scalar product defined by $w(t)$. See the Appendix for more details on Laguerre polynomials.

The Laguerre expansion of a continuous function y has a similar expression as in (1), (8) and its truncated approximation is then

$$y_N(t) = \sum_{n=0}^{N} \mu_n\,\ell_n^{(\alpha)}(t), \tag{14}$$

for some constant $N > 0$ and where

$$\mu_n = \frac{1}{\binom{n+\alpha}{n}\,\Gamma\,(\alpha+1)} \int_{-\infty}^{\infty} y(\tau)\,\ell_n(\tau)\,\tau^{\alpha-1}\,\mathrm{e}^{-\tau}\,d\tau.$$

Now, it follows from (14):

$$Y_N(s) = \sum_{n=0}^{N} \lambda_n\,L_n^{(\alpha)}(s), \tag{15}$$

where $L_n^{(\alpha)}$ the Laplace transform of $\ell_n^{(\alpha)}$. From the closed formula for Laguerre polynomials in (A1) (see the Appendix), it results that

$$L_n^{(\alpha)}(s) := \mathscr{L}\left(\ell_n^{(\alpha)}\right)(s) = \sum_{i=0}^{n} (-1)^i \binom{n+\alpha}{n-i} \frac{1}{s^{i+1}}.$$

Eliminating denominators in (15) provides

$$\begin{aligned} s^{N+1} Y_N(s) = {}& \lambda_N \left((-1)^N + \sum_{k=1}^{N-1} (-1)^k \binom{n+\alpha}{n-k} s^{N-k} \right) \\ & + \sum_{n=0}^{N-1} \lambda_n s^{N+1} L_n^{(\alpha)}(s). \end{aligned} \tag{16}$$

As in the case of Hermite polynomials, we proceed to the identification of the dominant coefficient λ_N and fix $\Theta_{\text{est}} = \{\lambda_N\}$. Similarly to (11), there is a relation $(\mathscr{R})$:

$$(\mathscr{R}) \qquad P\,(s)\,Y_N(s) + Q(s) + \overline{Q}(s) = 0, \tag{17}$$

where $P(s) = s^{N+1}$, $Q(s) = -\lambda_N\,s^{N+1} L_N(s)$, $\overline{Q}(s) = -\sum_{n=0}^{N-1} \lambda_n\,s^{N+1} L_n(s)$.

In the case $\alpha = 0$ and $N = 3$, the corresponding polynomials are $P(s) = s^4$, $Q(s) = (-s^3 + 3s^2 - 3s + 1)\lambda_3$ and $\overline{Q}(s) = -s^3\,\lambda_0 - \left(-s^3 + s^2\right)\,\lambda_1 + \left(-s^3 + 2s^2 - s\right)$.

In the next section, we introduce useful concepts in the algebraic method, such as *annihilators* and *estimators*, as well properties used to design these differential operators.

Remark 3.1: The algebraic method proposed in this paper is inspired by the parameter estimation algebraic methods initiated some years ago, for instance, in Fliess and Sira-Ramírez (2003). Some works on the identification of amplitudes, phases and frequencies can be found in Ushirobira et al. (2012, 2013) and Ushirobira, Perruquetti, and Mboup (2016b) and the case of parameters of an ODE in Ushirobira and Quadrat (2016).

4. Annihilators

To simplify this section, we consider $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. In the Appendix or in McConnell and Robson (2000), the reader may find more details on the algebraic structural properties used here.

Let B denote the polynomial ring on $\frac{d}{ds}$ with coefficients in $\mathbb{K}\,(s)$ (s denotes the Laplace variable). It is well known that B is a left principal domain.

Definition 4.1: Let $R \in \mathbb{K}_\Theta[s]$. A *R-annihilator with respect to* B is an element of $\mathrm{Ann}_{\mathsf{B}}(R) = \{F \in \mathsf{B} \mid F\,(R) = 0\}$.

The ideal $\mathrm{Ann}_{\mathsf{B}}(R)$ is left principal ideal, that means it is generated by a unique $\Pi_{\min} \in \mathsf{B}$, up to multiplication by a nonzero polynomial in B. So $\mathrm{Ann}_{\mathsf{B}}(R) = \mathsf{B}\,\Pi_{\min}$. The operator $\Pi_{\min}$ is a *minimal Q-annihilator with respect to* B. Notice that $\mathrm{Ann}_{\mathsf{B}}(R)$

contains annihilators in finite integral form, i.e. differential operators with coefficients in $\mathbb{K}\left[\frac{1}{s}\right]$. The following lemmas are useful:

Lemma 4.1: *Consider $R = s^n$ for $n \in \mathbb{N}$. A minimal R-annihilator is given by $\Pi_n = s\,\frac{d}{ds} - n$.*

For $m, n \in \mathbb{N}$, the operators Π_m and Π_n commute and so the following Lemmas are valid:

Lemma 4.2: *Let $R_1, R_2 \in \mathbb{K}_\Theta[s]$. Let F_i be a R_i-annihilator for $i = 1, 2$, such that $F_1 F_2 = F_2 F_1$. Then, $F_1 F_2$ is a $(\mu R_1 + \eta R_2)$-annihilator for all $\mu, \eta \in \mathbb{K}_\Theta$.*

Lemma 4.3: *Let $R \in \mathbb{K}_\Theta[s]$. Then, a minimal R-annihilator with respect to $\mathsf{B}_{\Theta_{est}}$ is given by $\Pi_{\min} = R\,\frac{d}{ds} - \frac{dR}{ds}$.*

To avoid the complete elimination by a $\overline{Q}$-annihilator of all terms in the relation $\mathscr{R}$ (see (3.1)) (hence of all parameters to be estimated), it is necessary to define an estimator.

Definition 4.2: An *estimator* $\Pi \in \mathsf{B}$ is a $\overline{Q}$-annihilator satisfying coeffs $(\Pi\,(\mathscr{R})) \cap \mathbb{K}_\Theta = \emptyset$, where coeffs($\cdot$) denotes the coefficients of a given polynomial.

For instance, from (13), we have

$$\overline{Q}(s) = -s^2\,\lambda_0 - 2\,s\,\lambda_1 \text{ and } Q(s) = 2\,\left(s^2 - 4\right)\lambda_2.$$

The differential operator $\Pi = \frac{d^3}{ds^3}$ is clearly an $\overline{Q}$-annihilator (since the degree of $\overline{Q}$ in s is equal to 2). However, the action of Π on $(\mathscr{R})$ defined by (12) also annihilates Q, which is the polynomial containing the parameter to be identified. Hence, Π is not an estimator.

From Lemma 4.1, it follows a minimal $\overline{Q}$-annihilator for s^2:

$$\pi_1 = s\,\frac{d}{ds} - 2.$$

Next, to complete eliminate $\overline{Q}$, we can apply $\pi_2 = s\,\frac{d}{ds} - 1$ on $\pi_1\left(\overline{Q}\right)$. That gives a $\overline{Q}$-annihilator written in the canonical form (see the Appendix):

$$\pi = \pi_2\,\pi_1 = s^2\,\frac{d^2}{ds^2} - 2\,\frac{d}{ds} + 2$$

which is also an estimator since $\pi(Q) \neq 0$ as we will analyse in the following section.

5. Examples

5.1 Example 1: Hermite expansion series of x

To illustrate our method, in this subsection, we work with a truncate Hermite series expansion of order 2 and estimating λ_2.

From (12), we have

$$P(s) = s^3,\;\; Q(s) = \left(2\,s^2 - 8\right)\,\lambda_2,\;\; \overline{Q}(s) = -\lambda_0\,s^2 - 2\,\lambda_1\,s.$$

We begin by eliminating the highest degree term in s in $\overline{Q}$, that is the term that contains λ_0. As we have just seen, Lemma 4.1 provides two annihilators π_1 and π_2 whose product is

$$\pi = \pi_2\,\pi_1 = s^2\,\frac{d^2}{ds^2} - 2\,\frac{d}{ds} + 2$$

written in the canonical form (A2). So, π annihilates $\overline{Q}$, but its action on Q is given by

$$\pi\,(Q) = -16\lambda_2.$$

Applying π on (12) yields

$$s^3\,\left(2 + 4\,s\frac{\mathrm{d}}{\mathrm{d}s} + s^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right)Y_N\,(s) - 16\,\lambda_2 = 0. \tag{18}$$

With the help of the inverse Laplace transform, the above equation can be brought to the time domain. For that, we multiply (18) by $\frac{1}{s^4}$ and apply the transform (5) giving

$$\int_0^t (t-\tau)^2\,y\,(\tau)\;\mathrm{d}\tau - 4\int_0^t \tau\;(t-\tau)\,y\,(\tau)\;\mathrm{d}\tau$$
$$+\int_0^t \tau^2 y\,(\tau)\;\mathrm{d}\tau - \frac{2}{15}\,\lambda_2\,t^5 = 0. \tag{19}$$

Solving (19) with respect to λ_2 gives

$$\lambda_2 = \frac{1}{t^5}\int_0^t \left(\upsilon_{2,0} - 4\,\upsilon_{1,1} + \upsilon_0\right)\;y\,(\tau)\;\mathrm{d}\tau \tag{20}$$

where the notation in (6) is used.

We illustrate this example with the signal $y(t) = \cos(2t) + \varpi(t)$ where the noise is $\varpi(t) = 2.10^2.\sin(10^{10}\,t)$. Its Hermite expansion series truncated at $N = 2$ is: $y(t) \approx 0.3678794412 - 0.1839397206\,h_2(t)$. From (20), an estimate of λ_2 is obtained. Then, by Theorem 2.1, the second derivative of y can be estimated. In Figure 1, we show the comparison between this estimate of $\widehat{\ddot{y}}(0)$ and the real value of $\ddot{y}(0)$ which is -4.

The coefficient λ_3 was estimated in Ushirobira and Quadrat (2016) using the present algebraic method.

5.2 Example 2: Laguerre expansion series of x

In the case of generalised Laguerre polynomials, we illustrate in this subsection how to estimate λ_3 in a truncated series expansion of order 3 as in (14), using the proposed algebraic technique. In this sense, we consider the relation $(\mathscr{R})$ obtained in (17) when $\alpha = 0$ and $N = 3$:

$$\begin{cases} P(s) = s^4, \\ Q(s) = \left(-s^3 + 3s^2 - 3s + 1\right)\lambda_3, \\ \overline{Q}(s) = -s^3\,\lambda_0 - \left(-s^3 + s^2\right)\,\lambda_1 + \left(-s^3 + 2s^2 - s\right). \end{cases} \tag{21}$$

We start the elimination of $\overline{Q}$ by annihilating λ_0. For that, Lemma 4.1 is used to obtain $\pi_1 = s\,\frac{d}{ds} - 3$ whose action on $\overline{Q}$ and Q results

$$\pi_1\left(\overline{Q}\right) = -s^2\lambda_1 + \left(-2\,s^2 + 2\,s\right)\lambda_2,$$
$$\pi_1\,(Q) = \left(-3\,s^2 + 6\,s - 3\right)\lambda_3.$$

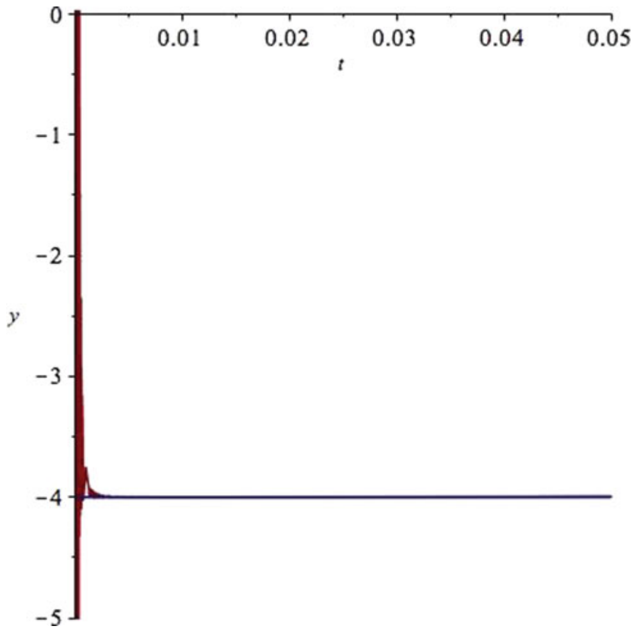

Figure 1. Comparison between $\widehat{\ddot{y}}(0)$ and $\ddot{y}(0)$.

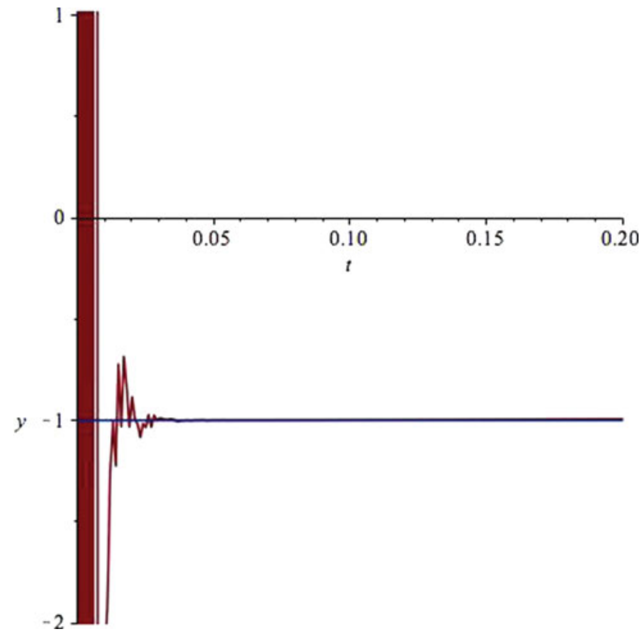

Figure 2. Comparison between $\widehat{y^{(3)}}(0)$ and $y^{(3)}(0)$.

Since the degree of $\pi_1\left(\overline{Q}\right)$ is 2, using Lemma 4.1 once more, we apply $\pi_2 = s\,\frac{d}{ds} - 2$ on $\pi_1\left(\overline{Q}\right)$ and $\pi_1(Q)$. That gives

$$\pi_2\left(\pi_1\left(\overline{Q}\right)\right) = -2\,s\lambda_2, \quad \pi_2\left(\pi_1\left(Q\right)\right) = -6\ (s-1)\ \lambda_3.$$

One last time, Lemma 4.1 helps to find $\pi_3 = s\,\frac{d}{ds} - 1$ to completely annihilate $\overline{Q}$. The resulting $\overline{Q}$-annihilator $\pi = \pi_3\,\pi_2\,\pi_1$ is also an estimator since $\pi(Q) = -6\lambda_3 \neq 0$. Moreover, it can be rewritten in the canonical form (A2) as follows:

$$\pi = s^3\,\frac{d^3}{ds^3} - 3\,s^2\,\frac{d^2}{ds^2} + 6\,s\,\frac{d}{ds} - 6.$$

We apply π on (16) and it follows

$$-6\,\lambda_3 + \left(18\,s^5\,\frac{\mathrm{d}}{\mathrm{d}s} + 9\,s^6\,\frac{\mathrm{d}^2}{\mathrm{d}s^2} + s^7\,\frac{\mathrm{d}^3}{\mathrm{d}s^3} + 6\,s^4\right) Y_N\,(s) = 0.$$

The inverse Laplace transform (5) helps to return to the time domain:

$$-\frac{\lambda_3\,t^7}{840} - 9\int_0^t \tau\,(t-\tau)^2\,y\,(\tau)\,\mathrm{d}\tau + 9\int_0^t \tau^2\,(t-\tau)\,y\,(\tau)\,\mathrm{d}\tau - \int_0^t \tau^3 y\,(\tau)\,\mathrm{d}\tau + \int_0^t (t-\tau)^3\,y\,(\tau)\,\mathrm{d}\tau = 0. \tag{22}$$

Finally, solving (22) with respect to λ_3 and using the notation (6), we obtain

$$\lambda_3 = -\frac{840}{t^7}\int_0^t \left(9\upsilon_{2,1} - 9\upsilon_{1,2} + \upsilon_{0,3} - \upsilon_{3,0}\right) y(\tau)\ d\tau. \tag{23}$$

This example is illustrated with the signal $y(t) = \sin(t) + \varpi(t)$ where the noise is $\varpi(t) = 2.10^2.\sin(10^{10}\,t)$. Its Laguerre expansion series truncated at $N = 3$ is

$$y(t) \approx 0.5\,\ell_0(t) + 4.10^{-22}\,\ell_1(t) - 0.25\,\ell_2(t) - 0.25\,\ell_3(t).$$

From (23), we obtain an estimate of λ_3. Hence, Theorem 2.1 states that the third derivative of y can be estimated from it. In Figure 2, we show the comparison between this estimate of $\widehat{y^{(3)}}(0)$ and the real value of $y^{(3)}(0)$ which is -1.

6. Conclusion

Numerical derivation is an important issue often arising in several problems in signal processing and control engineering. The approach presented in this paper is of algebraic flavour and it is based on the orthogonal polynomials series expansion of a given signal. Estimation of its derivatives is deduced from the estimates of the series expansion coefficients. The computations are performed in the Laplace operational domain, so using algebraic expressions. closed-form estimates are obtained, thanks to structural algebraic properties of the differential operators acting on these expressions. The choice of the differential operators is a critical step in the algebraic method proposed here; it permits a better-posed problem, and, so, better estimates. In addition, the integrals providing the estimation serve as filters for the noise in the signal. The cases of Hermite and Laguerre orthogonal polynomials were chosen to demonstrate our algebraic method. Two examples are given to illustrate our approach. Future work should include errors analysis, notably from the truncation of the series expansion.

Note

1. The noise here is interpreted as a *fast oscillation* and it does not depend on any probabilistic modeling, as in (Fliess 2006, 2008).

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Appendix

A.1. Classical orthogonal polynomials

The definition of some classical orthogonal polynomials is recalled in this part. General properties of Hermite and Laguerre polynomials are provided. The reader may check for more details at Abramowitz and Stegun (1964) or Szegö (1975).

A.1.1. Hermite polynomials. The definition of Hermite polynomials is given by

$$h_n(t) = (-1)^n \, \mathrm{e}^{t^2} \, \frac{d^n}{dt^n} \, \mathrm{e}^{-t^2} .$$

Hermite polynomials of even degree are even functions and those of odd degree are odd functions. Thus, we can write

$$h_n(t) = 2^n \, t^n + he_n(t),$$

where $he_n(t)$ is a polynomial with non-zero coefficients for all even powers of t smaller than n if n is even and for all odd powers if n is odd. Hermite polynomials are orthogonal with respect to the scalar product defined by the weight function $w(t) = \mathrm{e}^{-t^2}$:

$$\langle h_n(t), h_m(t) \rangle = \int_{-\infty}^{\infty} h_m(\tau) \, h_n(\tau) \, w(\tau) \, d\tau = \sqrt{\pi} \, 2^n \, n \, \delta_{mn}.$$

A.1.2. (Generalised) Laguerre polynomials. An explicit representation of generalised Laguerre polynomials is given by

$$\ell_n^{(\alpha)}(t) = \sum_{i=0}^{n} \binom{n+\alpha}{n+i} \frac{(-t)^i}{i!}. \tag{A1}$$

The analogue of Rodrigues' formula is

$$e^{-t}\, t^{\alpha}\, \ell_n^{(\alpha)}(t) = \frac{1}{n!} \frac{d^n}{dt^n} \left(e^{-t}\, t^{n+\alpha} \right).$$

They are orthogonal with respect to the weight function $w(t) = e^{-t}\, t^{\alpha}$, the scalar product defined by

$$\langle \ell_n^{(\alpha)}(t), \ell_m^{(\alpha)}(t) \rangle = \int_{-\infty}^{\infty} \ell_n^{(\alpha)}(\tau)\, \ell_m^{(\alpha)}(\tau)\, w(\tau)\, d\tau = 0, \ m \neq n.$$

A.2. The Weyl Algebra: basic notions

Definition A.1: Let $\mathbb{K}$ be a field of characteristic zero. Let $k \in \mathbb{N} \setminus \{0\}$. The *Weyl algebra* $\mathsf{A}_k = \mathsf{A}_k(\mathbb{K})$ is the free $\mathbb{K}$-algebra generated by $p_1, q_1, \ldots, p_k, q_k$ satisfying the relations

$$1 \leq i, j \leq k, \quad [p_i,\ q_j] = \delta_{ij}, \quad [p_i,\ p_j] = [q_i,\ q_j] = 0,$$

where $[u, v] := uv - vu$ is the *commutator* defined by for all u, $v \in \mathsf{A}_k(\mathbb{K})$ and δ_{ij} is the *Kronecker function*, i.e. $\delta_{ij} = 1$, if $i = j$ and 0, if $i \neq j$.

The Weyl algebra A_k is commonly realised as the $\mathbb{K}$-algebra of polynomial differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ such that $p_i := \frac{\partial}{\partial s_i}$ is the derivative with respect to s_i and $q_i := s_i \times$ represent the multiplication operator, for $1 \leq i \leq k$.

As a consequence, we can write

$$\begin{aligned} \mathsf{A}_k &= \mathbb{K}[q_1, \ldots, q_k][p_1, \ldots, p_k] \\ &= \mathbb{K}[s_1, \ldots, s_k]\left[\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_k}\right]. \end{aligned}$$

Remark A.1: The same notation is used for the variable s_i and for the operator 'multiplication by s_i'.

A closely related algebra to $\mathsf{A}_k(\mathbb{K})$ is defined as the differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ with coefficients in the rational functions field $\mathbb{K}(s_1, \ldots, s_k)$. We denote it by $\mathsf{B}_k(\mathbb{K})$, or B_k for short. We can write

$$\begin{aligned} \mathsf{B}_k &:= \mathbb{K}(q_1, \ldots, q_k)[p_1, \ldots, p_k] \\ &= \mathbb{K}(s_1, \ldots, s_k)\left[\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_k}\right]. \end{aligned}$$

Proposition A.1: *A basis for A_k is given by $\{q^I p^J \mid I, J \in \mathbb{N}^k\}$, where $q^I := q_1^{i_1} \ldots q_k^{i_k}$ and $p^J := p_1^{j_1} \ldots p_k^{j_k}$ if $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$. So an operator $F \in \mathsf{A}_k$ can be written in a canonical form*

$$F = \sum_{I,J} \lambda_{IJ} q^I p^J \ \text{ with } \lambda_{IJ} \in \mathbb{K}. \tag{A2}$$

Example A.1: We need later the following useful identity:

$$p^n q^m = q^m p^n + \sum_{k=1}^{n} \binom{n}{i}\binom{m}{i} i! q^{m-i} p^{n-i}$$

An element $F \in \mathsf{B}_k$ can be similarly written in a *canonical form*:

$$F = \sum_{I} \lambda_I\, G_I(s)\, p^I, \quad \text{where } G_I(s) \in \mathbb{K}(s_1, \ldots, s_k).$$

The *order* of an element $F \in \mathsf{B}_k$, $F = \sum_I G_I(s)\, p^I$ is defined as $\operatorname{ord}(F) := \max\{[I| \mid G_I(s) \neq 0\}$. The same definition holds for the Weyl algebra A_k since $\mathsf{A}_k \subset \mathsf{B}_k$. Some properties of A_k and B_k are given by the following propositions:

Proposition A.2: *The algebra A_k is a domain. Moreover, A_k is a simple algebra (i.e. it contains no nontrivial ideals) and also a left Noetherian ring (i.e. every left ideal is finitely generated). These properties are shared by B_k.*

Furthermore, A_k is neither a principal right domain, nor a principal left domain, while this is true for B_k:

Proposition A.3: *B_1 admits a left division algorithm, that is, if $F, G \in \mathsf{B}_1$, then there exists Q, $R \in \mathsf{B}_1$ such that $F = QG + R$ and* $\operatorname{ord}(R) < \operatorname{ord}(G)$. *So B_1 is a principal left domain.*