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Sensitivity analysis using Itô-Malliavin calculus and martingales. Application to stochastic optimal control.

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SENSITIVITY ANALYSIS USING ITÔ-MALLIAVIN CALCULUS AND MARTINGALES. APPLICATION TO STOCHASTIC OPTIMAL CONTROL.

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Abstract. We consider a multidimensional diffusion process $(X_t^\alpha)_{0\leq t\leq T}$ whose dynamics depends on parameters α. Our first purpose is to give representation formulae of the sensitivity $\nabla_\alpha J(\alpha)$ for the expected cost $J(\alpha)=\mathbb{E}(f(X_T^\alpha))$ as an expectation: this issue is motivated by stochastic control problems (where the controller is parameterized and the optimization problem is then reduced to a parametric optimization one) or by model misspecifications in finance. Known results concerning the evaluation of $\nabla_\alpha J(\alpha)$ by simulations concern the case of smooth cost functions f or of diffusion coefficients not depending on α (see Kushner and Yang, *SIAM J. Control Optim. 29 (5)*, pp. 1216-1249, 1991). Here, we handle the general case removing these two restrictions, deriving three new type formulae to evaluate $\nabla_\alpha J(\alpha)$: we call them *Malliavin calculus approach*, *adjoint approach* and *martingale approach*. For this, our basic tools are Itô calculus, Malliavin calculus and martingale arguments. In the second part of this work, we provide discretization procedures to simulate the relevant random variables and we analyze the associated weak error: the nature of the results are new in that context. We prove that the discretization error is essentially linear w.r.t. the time step. Finally, some numerical experiments deal with some examples in random mechanics and finance: we compare different methods in terms of variance, complexity, computational time and influence of the time discretization step.

Key words. Sensitivity analysis, Malliavin calculus, weak approximation.

AMS subject classifications. 90C31, 93E20, 60H30

1. Introduction. We consider a multidimensional stochastic differential equation (SDE in short) defined by

$$X_t = x + \int_0^t b(s,X_s,\alpha)\,ds + \sum_{j=1}^q \int_0^t \sigma_j(s,X_s,\alpha)\,dW_s^j. \tag{1.1}$$

where α is a parameter (taking values in $\mathcal{A}\subset\mathbb{R}^m$) and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion in $\mathbb{R}^q$ on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathbb{P})$, with the usual assumptions on the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$.

We are interested in studying how to evaluate the sensitivity w.r.t. α of the expected cost

$$J(\alpha)=\mathbb{E}\left(f(X_T)\right), \tag{1.2}$$

for a given terminal cost f and for a fixed time T. The sensitivity of more general functional including instantaneous costs like $\mathbb{E}\left(\int_0^T g(t,X_t)dt + f(X_T)\right)$ may follow by linearity.

This question is raised in many applications. Indeed, this is a classical issue when we need to analyze the impact of a misspecification of some stochastic model (defined

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by a SDE with coefficients $\bar{b}(t,x)$ and $(\bar{\sigma}_j(t,x))_{1\leq j\leq q})$ on the expected cost $J(\alpha)$; it may formalized in setting $b(t,x,\alpha) = \bar{b}(t,x) + \sum_{i=1}^m \alpha_i\phi_i(t,x)$ (and analogously for $(\sigma_j(t,x,\alpha))_{1\leq j\leq q})$ and the sensitivities are computed at the point $\alpha = 0$. In finance, this is the so-called model risk problem (see Cvitanic and Karatzas [CK99]) and misspecifications usually concern the diffusion coefficients $(\bar{\sigma}_j(t,x))_{1\leq j\leq q}$.

An other class of problems concerned by the sensitivity analysis are stochastic control problems. If the controlled SDE is defined by $dX_t = \bar{b}(t,X_t,u_t)\ dt + \sum_{j=1}^q \bar{\sigma}_j(t,X_t,u_t)dW_t^j$, we may seek the optimal policy $(u_t)_{0\leq t\leq T}$ (for the maximization problem of $\mathbb{E}(f(X_T))$ for instance) in a feedback form using a parametric approach, that is $u_t = u(t,X_t,\alpha)$: in that case, one puts $b(t,x,\alpha) = \bar{b}(t,x,u(t,x,\alpha))$ and $\sigma_j(t,x,\alpha) = \bar{\sigma}_j(t,x,u(t,x,\alpha))$. The policy function $u(t,x,\alpha)$ can be parameterized through a linear approximation (linear combination of basis functions) or in a non linear way (like neural networks, see Rumelhart and McClelland [RM86] or Haykin [Hay94] for general references). Thus, a standard parametric optimization procedure such as stochastic gradient methods or other stochastic approximation algorithms (see Poljak [Pol87]; Benveniste, Metivier and Priouret [BMP90]; Kushner and Yin [KY97]) may be used and for this, some sensitivity estimations of $J(\alpha)$ w.r.t. α, like $\nabla_\alpha J(\alpha)$, are needed. This gradient is the quantity that we will focus on in this paper.

Since the setting is a priori multidimensional, we privilege a Monte Carlo approach for the numerical computations. The evaluation of $J(\alpha)$ is standard and has been widely studied: we refer the reader to Kloeden and Platen [KP95] for instance, for an introduction to numerical approximations of SDEs. Concerning the computations of $\nabla_\alpha J(\alpha)$ in our context, there exist three different approaches to our knowledge.

1. We may use the *re-simulation method* (see Glasserman and Yao [GY92], L'Ecuyer and Perron [LP94] for instance): it consists in computing different values of $J(\alpha)$ for some close values of the parameter α, and then forming some appropriate differences to approximate the derivatives. However, it turns out to be costly if the dimension of the parameter α is large and moreover, it provides biased estimators.
2. The *path-wise method* (proposed in our context by Kushner and Yang [KY91]) consists in putting the gradient inside the expectation, involving ∇f and $\nabla_\alpha X_T$: $\nabla_\alpha J(\alpha)$ is thus expressed as an expectation (see Proposition 1.1 below) and Monte Carlo methods can be used. One limitation of this method is that the cost function f has to be smooth.
3. An other alternative method is the so-called *likelihood method* or *score method* (introduced by Glynn [Gly86, Gly87], Reiman and Weiss [RW86]; see also Broadie and Glasserman [BG96] for applications to the computation of Greeks in finance), where the gradient is rewritten as $\mathbb{E}(f(X_T)H)$ for some random variable H. There is no uniqueness in this representation, since we can add to H any random variables orthogonal to X_T. As for the path-wise method, the estimator is unbiased, but as a difference, the cost function f needs not to be smooth. Usually, H is equal to $\nabla_\alpha(\log(p(\alpha,X_T)))$ where $p(\alpha,.)$ is the density w.r.t. the Lebesgue measure of the law of X_T. It has some strong limitations in our context since in general, this quantity is unknown. However, Kushner and Yang [KY91] provide some explicit weights H, under the restriction that α concerns only b (and not σ_j) and that the diffusion coefficient is elliptic, using the Girsanov theorem (see Proposition 2.6).

A **first objective** of this work is to handle more general situations where both coefficients defining the SDE (1.1) depend on α. To this issue, we provide three new answers to express the sensitivity of $J(\alpha)$ with respect α.

1. The first contribution is rather an extension of the likelihood approach method to the case of diffusion coefficients depending on α. It uses a direct integration by part formula of the Malliavin calculus: this kind of idea has been quite recently used in a financial context in the paper by Fournié *et al.* [FLL+99], to compute some sensitivities of option prices. These techniques have also been efficiently used by the first author to derive asymptotic properties of statistical procedures when we estimate parameters defining a SDE (see [Gob01b, Gob02]). Actually, our true contribution concerns essentially a situation where ellipticity is replaced by a weaker (but standard) non degeneracy condition, which enables to deal with random mechanics problems or portfolio optimization problems in finance.
2. The second approach is rather different from previous methods. Namely, we initially focus on the adjoint point of view (see Bensoussan [Ben88] or Peng [Pen90]), to finally derive new formulae, involving again some integration by part formula, but written in a simple way (using only Itô's calculus). In stochastic control problems, adjoint processes are related to Backward SDEs (see Yong and Zhou [YZ99] e.g.) and their simulations turn out to be a difficult task. Here, we circumvent this difficulty since we only need to express them as explicit conditional expectations, which is feasible.
3. The third approach follows from martingale arguments applied to the expected cost and leads to an original representation, which turns out to be surprisingly simple.

What helps to compare these new methods together with the previous ones is the variance of the random variables involved in the resulting formulae for $\nabla_\alpha J(\alpha)$: this is numerically studied in section 5.

An other element of comparison is the influence of time step h, which is used to approximately simulate the random variables. The analysis of these discretization errors is the **second** significant **part** of this work. The random variables which are concerned are essentially written as $f(X_T)H$ and simulations are based on Euler schemes: although H has a complex form, we first explicit an algorithm of approximation and then analyze the induced error w.r.t. the time step h. This part of the paper is also original, since to our knowledge, results in the literature only concern the approximation of $\mathbb{E}(f(X_T))$.

Outline of the paper. In the following, we define the notations and some assumptions which will be used throughout the paper. We also recall the *path-wise* approach in Proposition 1.1. In section 2, after giving some standard facts on the Malliavin calculus, we develop our three approaches to compute the sensitivity of $J(\alpha)$ w.r.t. α: these are the so-called *Malliavin calculus* approach (Propositions 2.5 and 2.8), the *adjoint* approach (Theorem 2.11) and the *martingale* approach (Theorem 2.12). In section 3, we provide simulation schemes to compute $\nabla_\alpha J(\alpha)$ by an usual Monte-Carlo approach, using the methods developed before: the quantity to adjust is the time step h used for Euler type schemes and we analyze its influence in each method. We reserve for these analyzes a significant place in the paper since these problems have not yet been handled in the literature. The approximation results are stated in Theorems 3.1, 3.2, 3.4 and 3.5, while their proofs are postponed in section 4. Finally, numerical experiments in section 5 illustrate the methods developed: we compare the computational time, the complexity, the variance of the estimators on examples borrowed to finance and control problems.

Assumptions. In the mentioned applications, the parameter is *a priori* multidimensional but since in the following we will look at sensitivities w.r.t. α coordinate-wise, this is not a restriction to assume that $\mathcal{A} \subset \mathbb{R}$ ($m = 1$).

The process defined in (1.1) depends on the parameter α, but we omit this dependence in the notation. Furthermore, the initial condition $X_0 = x$ is fixed throughout the paper. We denote σ_j the j-th column vector of σ.

To study the sensitivity of J (defined in (1.2)) w.r.t. α, we may assume in the sequel that coefficients are enough smooth.

Assumption (R): the functions b and σ are of class C^∞ w.r.t. the variables t, x, α, with bounded derivatives.

Note that b and σ may be unbounded. At several places, the diffusion coefficient will be required to be uniformly elliptic, in the following sense.

Assumption (E): σ is a squared matrix ($q = d$) such that the matrix $\sigma\sigma^*$ satisfies an uniform ellipticity condition:

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \qquad [\sigma\sigma^*](t, x, \alpha) \geq \mu_{min}\, \mathrm{I}_d$$

for some real number $\mu_{min} > 0$.

Notation.

- *Differentiation.* As usual, derivatives w.r.t. α will be simply denoted with a dot, that is $\partial_\alpha J = \dot{J}$ for instance. If no ambiguity is possible, we will omit to write explicitly the parameter α in b, $\sigma_j \cdots$. We adopt the following usual convention on the gradients: if $\psi : \mathbb{R}^{p_2} \mapsto \mathbb{R}^{p_1}$ is a differentiable function, its gradient $\nabla_x \psi(x) = (\partial_{x_1}\psi(x), \cdots, \partial_{x_{p_2}}\psi(x))$ takes values in $\mathbb{R}^{p_1} \otimes \mathbb{R}^{p_2}$. At many places, $\nabla_x \psi(x)$ will be simply denoted $\psi'(x)$.
- *Linear algebra.* The r-th column of a matrix A will be denoted by A_r (or $A_{r,t}$ if A is a time dependent matrix) and the r-th element of a vector a will be denoted by a_r (or $a_{r,t}$ if a is a time dependent vector). A^* stands for the transpose of A.
  For a matrix A, the matrix obtained by keeping only the last r rows (resp. the last r columns) will be denoted $\Pi_r^{Row}(A)$ (resp. $\Pi_r^{Col.}(A)$).
  For $i \in \{1, \cdots, d\}$, we set $e^i = (0 \cdots 0\ 1\ 0 \cdots 0)^*$, where 1 is the i-th coordinate.
- *Constants.* We will keep the same notation $K(T)$ for all finite, non-negative and non-decreasing functions: they do not depend on x, the function f or further discretization steps h but they may depend on the coefficients $b(.)$ and $\sigma(.)$. The generic notation $K(x, T)$ stands for any function bounded by $K(T)(1 + |x|^Q)$ for some $Q \geq 0$.

When a function $g(s, x, \alpha)$ is evaluated at $x = X_s^\alpha$, we may sometimes use the short notation g_s if no ambiguity is possible. For instance, (1.1) may be written as $X_t = x + \int_0^t b_s ds + \sum_{j=1}^q \int_0^t \sigma_{j,s} dW_s^j$.

Other processes related to $(X_t)_{0 \leq t \leq T}$. To the diffusion X under **(R)**, we can associate its flow, i.e. the Jacobian matrix $Y_t := \nabla_x X_t$, the inverse of its flow $Z_t = {Y_t}^{-1}$ and the path-wise derivative of X_t with respect to α which we denote $\dot{X}_t$

(see Kunita [Kun84]). These processes solve

$$Y_t = \mathrm{I}_d + \int_0^t b'_s\, Y_s\, ds + \sum_{j=1}^q \int_0^t \sigma'_{j,s}\, Y_s\, dW_s^j, \tag{1.3}$$

$$Z_t = \mathrm{I}_d - \int_0^t Z_s(b'_s - \sum_{j=1}^q (\sigma'_{j,s})^2)\, ds - \sum_{j=1}^q \int_0^t Z_s \sigma'_{j,s}\, dW_s^j, \tag{1.4}$$

$$\dot{X}_t = \int_0^t \left(\dot{b}_s + b'_s\, \dot{X}_s\right) ds + \sum_{j=1}^q \int_0^t \left(\dot{\sigma}_{j,s} + \sigma'_{j,s}\, \dot{X}_s\right) dW_s^j. \tag{1.5}$$

Actually, since the process $(\dot{X}_t)_{0\leq t\leq T}$ satisfies a linear equation, it can also simply be written using Y_t and Z_t (apply Theorem 56 p.271 from Protter [Pro90]):

$$\dot{X}_t = Y_t \int_0^t Z_s\big[(\dot{b}_s - \sum_{j=1}^q \sigma'_{j,s}\dot{\sigma}_{j,s})ds + \sum_{j=1}^q \dot{\sigma}_{j,s} dW_s^j\big]. \tag{1.6}$$

If f is continuously differentiable with some appropriate growth condition (enough to apply the Lebesgue differentiation theorem), one immediately obtains the following result (see also Kushner and Yang [KY91]): we call it the *path-wise approach.*

Proposition 1.1. *Assume* **(R)**. *One has* $\dot{J}(\alpha) = \mathbb{E}\big(H_T^{Path.}\big)$ *with* $H_T^{Path.} = f'(X_T)\dot{X}_T$.

Hence, the gradient can be written still as an expectation, which is crucial for a Monte Carlo evaluation. One purpose of the paper is to extent this result to the case of non differentiable functions, by essentially writing $\dot{J}(\alpha) = \mathbb{E}(f(X_T)H)$ for some random variable H.

In the sequel, two types of assumption on f will be considered.

Assumption (H): f is a bounded measurable function.

Assumption (H'): f is a bounded measurable function and satisfies the following continuity estimate for some $p_0 > 1$

$$\int_0^T \frac{\|f(X_T) - f(X_t)\|_{\mathbf{L}^{p_0}}}{T-t}\, dt < +\infty.$$

This $\mathbf{L}^p$-smoothness assumption of $f(X_T) - f(X_t)$ is obviously satisfied for uniformly Hölder function with exponent β, but also for some non smooth function, such as the indicator function of a domain.

Proposition 1.2. *Let D be a domain of $\mathbb{R}^d$: suppose that either it has a compact and smooth boundary (say of class C^2, see [GT77]), or it is a convex polyhedron ($D = \cap_{i=1}^I D_i$ where $(D_i)_{1\leq i\leq I}$ are half-spaces). Assume* **(E)**, **(R)** *and bounded coefficients b and σ. Then, the function $f = \mathbf{1}_D$ satisfies the assumption* **(H')** *(for any $p_0 > 1$).*

Proof. Since $\|f(X_T) - f(X_t)\|_{\mathbf{L}^p}^p \leq \mathbb{E}|\mathbf{1}_D(X_T) - \mathbf{1}_D(X_t)| \leq \mathbb{P}(X_T \in D, X_t \notin D) + \mathbb{P}(X_T \notin D, X_t \in D)$, we only need to prove that $\mathbb{P}(X_T \in D, X_t \notin D) \leq K(T)(T-t)^\beta$ for some $\beta > 0$. Now, recall the standard exponential inequality $\mathbb{P}(\|X_u - x\| \geq \delta) \leq K(T)\exp(-c\frac{\delta^2}{u})$ (with $c > 0$) available for $u \in]0, T]$ and $\delta \geq 0$ (see e.g. Lemma 4.1 in [Gob00]). Combining this with the Markov property, it follows that $\mathbb{P}(X_T \in D, X_t \notin D) \leq K(T)\mathbb{E}(\mathbf{1}_{X_t \notin D} \exp(-c\frac{d^2(X_t, D^c)}{(T-t)})$. Then, a direct estimation of the above expectation using in particular a Gaussian upper bound for the density of the law of X_t (see Friedman [Fri64]) yields easily the required estimate with $\beta = \frac{1}{2}$ (see Lemma 2.8 in [Gob01a] for details). □

2. Sensitivity formulae. In this section, we present three different approaches to evaluate $\dot{J}(\alpha)$. Before this, we introduce the Malliavin calculus material necessary to our computations.

2.1. Some basic results on the Malliavin calculus. The reader may refer to Nualart [Nua95] (section 2.2 for the case of diffusion processes) for a detailed exposition of this section.

Put $\mathbf{H} = \mathbf{L}^2([0,T], \mathbb{R}^q)$: we will consider elements of $\mathbf{H}$ written as a row vector. For $h(.) \in \mathbf{H}$, denote by $W(h)$ the Wiener stochastic integral $\int_0^T h(t)\, dW_t$.

Let $\mathcal{S}$ denote the class of random variables of the form $F = f(W(h_1), \ldots, W(h_N))$ where $f \in C_p^\infty(\mathbb{R}^N)$, $(h_1, \ldots, h_N) \in \mathbf{H}^N$ and $N \geq 1$. For $F \in \mathcal{S}$, we define its derivative $\mathcal{D}F = (\mathcal{D}_t F := (\mathcal{D}_t^1 F, \cdots, \mathcal{D}_t^q F))_{t\in[0,T]}$ as the $\mathbf{H}$-valued random variable given by $\mathcal{D}_t F = \sum_{i=1}^N \partial_{x_i} f(W(h_1), \ldots, W(h_N))\, h_i(t)$. The operator $\mathcal{D}$ is closable as an operator from $\mathbf{L}^p(\Omega)$ to $\mathbf{L}^p(\Omega, \mathbf{H})$, for any $p \geq 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|_{\mathbf{H}}^p)]^{1/p}$. We can define the iteration of the operator $\mathcal{D}$, in such a way that for a smooth random variable F, the derivative $\mathcal{D}^k F$ is a random variable with values on $\mathbf{H}^{\otimes k}$. As in the case $k = 1$, the operator $\mathcal{D}^k$ is closable from $S \subset \mathbf{L}^p(\Omega)$ into $\mathbf{L}^p(\Omega; \mathbf{H}^{\otimes k})$, $p \geq 1$. If we define the norm $\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\|\mathcal{D}^j F\|_{\mathbf{H}^{\otimes j}}^p)]^{1/p}$, we denote its domain by $\mathbb{D}^{k,p}$.

One has the chain rule property:

PROPOSITION 2.1. *Fix $p \geq 1$. For $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$ and $F = (F_1, \cdots, F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \geq 0$, one has*

$$\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F, \text{ with the notation } \mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_s F_1 \\ \vdots \\ \mathcal{D}_s F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce δ, the Skorohod integral, defined as the adjoint operator of $\mathcal{D}$:

DEFINITION 2.2. *δ is a linear operator on $\mathbf{L}^2([0,T] \times \Omega, \mathbb{R}^q)$ with values in $\mathbf{L}^2(\Omega)$ such that:*

1. *the domain of δ (denoted by $\mathrm{Dom}(\delta)$) is the set of processes $u \in \mathbf{L}^2([0,T] \times \Omega, \mathbb{R}^q)$ such that $\left|\mathbb{E}(\int_0^T \mathcal{D}_t F \cdot u_t\, dt)\right| \leq c(u)\, \|F\|_{\mathbf{L}^2}$ for any $F \in \mathbb{D}^{1,2}$.*
2. *if u belongs to $\mathrm{Dom}(\delta)$, then $\delta(u)$ is the element of $\mathbf{L}^2(\Omega)$ characterized by the integration by part formula*

$$\forall F \in \mathbb{D}^{1,2}, \qquad \mathbb{E}\big(F\, \delta(u)\big) = \mathbb{E}\big(\int_0^T \mathcal{D}_t F \cdot u_t\, dt\big). \tag{2.1}$$

In the following proposition, we sum up few properties of the Skorohod integral.

PROPOSITION 2.3.

1. *The space of weakly differentiable $\mathbf{H}$-valued variables $\mathbb{D}^{1,2}(\mathbf{H})$ belongs to $\mathrm{Dom}(\delta)$.*
2. *If u is an adapted process belonging to $\mathbf{L}^2([0,T] \times \Omega, \mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincides: $\delta(u) = \int_0^T u_t\, dW_t$.*
3. *If F belongs to $\mathbb{D}^{1,2}$, then for any $u \in \mathrm{Dom}(\delta)$ s.t. $\mathbb{E}(F^2 \int_0^T \|u_t\|^2\, dt) < +\infty$,*

one has

$$\delta(F\ u) = F\ \delta(u) - \int_0^T \mathcal{D}_t F \cdot u_t\ dt, \tag{2.2}$$

whenever the r.h.s. above belongs to $\mathbf{L}^2(\Omega)$.

Concerning the solution of SDEs, it is well-known that for any $t \geq 0$, the random variables X_t, Y_t, Z_t and $\dot{X}_t$ belong to $\mathbb{D}^\infty$ under **(R)**. Furthermore, one has the following estimates: for any $k \geq 1$, $\mathbb{E}\left(\sup_{0\leq t\leq T} \|\mathcal{D}_{r_1,\cdots,r_k} U_t\|^p\right) \leq K(T,x)$, for $U_t = X_t$, Y_t, Z_t and $\dot{X}_t$. Besides, $\mathcal{D}_s X_t$ is given by:

$$\mathcal{D}_s X_t = Y_t\ Z_s \sigma(s, X_s)\ \mathbf{1}_{s\leq t}. \tag{2.3}$$

At last, we recall some standard results related to the integration by part formulae. The Malliavin covariance matrix of a smooth random variable F is defined by

$$\gamma^F = \int_0^T \mathcal{D}_t F\ [\mathcal{D}_t F]^*\ dt. \tag{2.4}$$

Proposition 2.4. *Let F be a random variable in $\mathbb{D}^\infty$ such that* $\det(\gamma^F)$ *is a.s. positive and* $1/\det(\gamma^F) \in \cap_{p\geq 1}\mathbf{L}^p$, *$G$ belong to $\mathbb{D}^\infty$ and g be a smooth function with polynomial growth. Then, for any multi-index $\bar{\gamma}$, there exists a random variable $H_{\bar{\gamma}}(F,G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial^{\bar{\gamma}} g(F) G] = \mathbb{E}[g(F) H_{\bar{\gamma}}(F,G)],$$
$$\|H_{\bar{\gamma}}(F,G)\mathbf{1}_A\|_{\mathbf{L}^p} \leq C \|[\gamma^F]^{-1}\mathbf{1}_A\|^{p_1}_{\mathbf{L}^{q_1}} \|F\|^{p_2}_{k_2,q_2} \|G\|_{k_3,q_3},$$

for some constants $C, p_1, p_2, k_2, k_3, q_1, q_2, q_3$ depending on p and $\bar{\gamma}$. Here, the event A is arbitrary.

Proof. See Propositions 3.2.1 and 3.2.2 p.160-161 in Nualart [Nua98] when $A = \Omega$. For general events A, see Proposition 2.4 from Bally and Talay [BT96a]. □

The construction of $H_{\bar{\gamma}}(F,G)$ is based on equality (2.1) and involves iterated Skorohod integrals. There are several ways to obtain such a formula: anyhow, we do not really need to explicit them at this stage.

2.2. First approach: direct Malliavin calculus computations. Here, the leading idea is to start from Proposition 1.1 and apply results like Proposition 2.4 to get $\dot{J}(\alpha) = \mathbb{E}(f(X_T)H)$. Nevertheless, there are several ways to do this, depending on whether the diffusion coefficient is elliptic (see also [FLL+99] in that situation) or not.

2.2.1. Elliptic case. Consider first that the assumption **(E)** is fulfilled.

Proposition 2.5. *Assume* **(R)**, **(E)** *and* **(H)**. *One has* $\dot{J}(\alpha) = \mathbb{E}\big(H_T^{Mall.Ell.}\big)$ *with*

$$H_T^{Mall.Ell.} = \frac{1}{T} f(X_T) \delta\big([\sigma_\cdot^{-1}\ Y_\cdot\ Z_T\ \dot{X}_T]^*\big).$$

Proof. We can consider that f is smooth, the general case being derived by a density argument. Because of (2.3) and Assumption **(E)**, one remarks that $\mathcal{D}_t X_T$ is invertible for any $t \in [0,T]$: thus, for such t, using the chain rule (Proposition 2.1), one gets that $f'(X_T) = \mathcal{D}_t(f(X_T))\sigma_t^{-1}\ Y_t\ Z_T$. Integrating in time over $[0,T]$ and using Proposition 1.1, one gets that $\dot{J}(\alpha) = \frac{1}{T}\int_0^T dt\ \mathbb{E}(\mathcal{D}_t(f(X_T))\sigma_t^{-1}\ Y_t\ Z_T\ \dot{X}_T)$. An application of the relation (2.1) completes the proof of Proposition 2.5. □

When the parameter is involved only the drift coefficient, the laws of $(X_t)_{0\le t\le T}$ for two different values of α are equivalent owing the Girsanov theorem. Exploiting directly this possible change of measure, a simplified expression for $\dot{J}(\alpha)$ can be found: this is the *likelihood ratio method* or *score method* from Kushner and Yang [KY91].

PROPOSITION 2.6. *Assume* **(R)**, **(E)** *and* **(H)**. *Suppose that the parameter of interest* α *is not in the diffusion coefficient. Then, one has:*

$$\dot{J}(\alpha) = \mathbb{E}\left(f(X_T)\int_0^T [\sigma_t^{-1}\dot{b}_t]^* dW_t \right).$$

Proof. We justify it without the Girsanov theorem, rather exploiting the particular form of $\dot{X}_T$ given in (1.6). Indeed, $f'(X_T)\dot{X}_T = f'(X_T)Y_T\int_0^T Z_t\dot{b}_t dt = \int_0^T dt\ \mathcal{D}_t(f(X_T))[\sigma_t^{-1}\dot{b}_t]$, and the result follows using (2.1). □

2.2.2. General non degenerate case. There are many situations where the ellipticity Assumption **(E)** is too stringent and can not be fulfilled. To illustrate these facts, let us rewrite the SDE in the following way, splitting its structure in two parts:

$$dX_t = \begin{pmatrix} dS_t \\ dV_t \end{pmatrix} = \begin{pmatrix} b_S(t,X_t,\alpha) \\ b_V(t,X_t,\alpha) \end{pmatrix} dt + \begin{pmatrix} \sigma_S(t,X_t,\alpha) \\ \sigma_V(t,X_t,\alpha) \end{pmatrix} dW_t. \tag{2.5}$$

Here, $(S_t)_{t\ge 0}$ is $(d-r)$-dimensional and $(V_t)_{t\ge 0}$ r-dimensional. The cost function of interest may involve only the value of V_T: $J(\alpha) = \mathbb{E}(f(V_T))$. Note that considering $r = d$ reduces to the previous situation. We now give two examples which motivate the statement of Proposition 2.7 below.

a) In Random Mechanics (see Krée and Soize [KS86]), the pair position/velocity $dX_t = \begin{pmatrix} dx_t \\ dv_t \end{pmatrix} = \begin{pmatrix} v_t dt \\ \cdots \end{pmatrix}$ can not satisfied an ellipticity condition, but weaker assumptions as hypoellipticity are more realistic.

b) In finance, namely in portfolio optimization (for a recent review, see e.g. Runggaldier [Run02]), r usually equals 1: $(S_t)_{t\ge 0}$ describes the dynamic of the risky assets, while $(V_t)_{t\ge 0}$ is the wealth process, corresponding to the value a self-financed portfolio invested in the assets $(S_t)_{t\ge 0}$ with respect the strategy $(\xi_t = \{\xi_i(t,X_t) : 1 \le i \le d-1\})_{t\ge 0}$: $dV_t = \sum_{i=1}^{d-1}\xi_i(t,X_t)dS_{i,t}$ (see e.g. Karatzas and Shreve [KS98]). It is clear that the resulting diffusion coefficient for the whole process $X_t = \begin{pmatrix} S_t \\ V_t \end{pmatrix}$ can not satisfy an ellipticity condition. Nevertheless, requiring that the matrix $\sigma_V\sigma_V^*(t,x)$ satisfies an ellipticity type condition is not much demanding in that framework.

We set γ_T for the Malliavin covariance matrix of V_T: $\gamma_T = \int_0^T \mathcal{D}_t V_T\ [\mathcal{D}_t V_T]^*\ dt$. This allows to reformulate Assumption **(E)** in

Assumption (E'): $\det(\gamma_T)$ is *a.s.* positive and for any $p \ge 1$, one has

$$\|1/\det(\gamma_T)\|_{\mathbf{L}^p} < +\infty.$$

We now bring together classical results related to Assumption **(E')**.

PROPOSITION 2.7. *Assumption* **(E')** *is fulfilled in the following situations.*

a) Hypoelliptic case *(with $r = d$). The Lie algebra generated by the vector fields* $\partial_t + A_0(t,x) := \partial_t + \sum_{i=1}^d (b - \frac{1}{2}\sum_{j=1}^q \sigma'_j\sigma_j)_i(t,x)\partial_{x_i}$, $A_j(t,x) := \sum_{i=1}^d \sigma_{i,j}(t,x)\partial_{x_i}$ *for* $1 \le j \le q$ *spans* $\mathbb{R}^{d+1}$ *at the point* $(0, X_0)$*:*

$$\dim \operatorname{span} \operatorname{Lie}(\partial_t + A_0, A_j, 1 \le j \le q)(0, X_0) = d + 1.$$

b) Partially elliptic case *(with $r \ge 1$). For some real number $\mu_{min} > 0$, one has*

$$\forall x \in \mathbb{R}^d, \qquad [\sigma_V\sigma_V^*](T,x,\alpha) \ge \mu_{min}\,\mathrm{I}_d.$$

Proof. The statement a) is standard and we refer to Cattiaux and Mesnager [CM02] for a recent account on the subject. The statement b) is also classical: see for instance the arguments in Nualart [Nua98], p.158-159. □

Now, we are in position to give a sensitivity formula under **(E')**.

PROPOSITION 2.8. *Assume* **(R)**, **(E')** *and* **(H)**. *One has* $\dot J(\alpha) = \mathbb{E}\big(H_T^{NonDeg.}\big)$ *with*

$$H_T^{NonDeg.} = f(V_T)\,\delta\big(\dot V_T^*\gamma_T^{-1}\mathcal{D}_.V_T\big).$$

Proof. The Assumption **(E')** validates (see Nualart [Nua98] Proposition 3.2.1) the following computations, adapted from the ones used for Proposition 2.5. The chain rule property yields $f'(V_T) = \int_0^T \mathcal{D}_t(f(V_T))[\mathcal{D}_tV_T]^*\gamma_T^{-1}\,dt$, and thus $\mathbb{E}(f'(V_T)\dot V_T) = \mathbb{E}(\int_0^T \mathcal{D}_t(f(V_T))[\mathcal{D}_tV_T]^*\gamma_T^{-1}\dot V_T\,dt)$. Proposition 2.8 now follows from (2.1). □

Proposition 2.8 is also valid under **(E)** in the case $r = d$, but it turns out that the formula in Proposition 2.5 is a bit simpler to implement.

2.3. A second approach based on the adjoint point of view.

2.3.1. Other representation of the sensitivity of $J(\alpha)$. Set $u(t,x) = \mathbb{E}(f(X_T)|X_t = x)$: remark that $J(\alpha) = u(0, X_0)$. Under smoothness assumptions on b and σ and non-degeneracy hypothesis on the infinitesimal generator of $(X_t)_{t\ge 0}$, it is well-known (see Cattiaux and Mesnager [CM02]) that u is the smooth solution of the partial differential equation (PDE in short)

$$\begin{cases} \partial_t u(t,x) + \displaystyle\sum_{i=1}^d b_i(t,x)\partial_{x_i}u(t,x) + \frac{1}{2}\sum_{i,j=1}^d [\sigma\sigma^*]_{i,j}(t,x)\partial^2_{x_i,x_j}u(t,x) = 0 \ \text{ for } t < T \\ u(T,x) = f(x). \end{cases}$$

Our purpose is to give an other expression for $\dot J(\alpha)$ of Proposition 1.1. The idea is simple: it consists in differentiating formally the PDE above w.r.t. α and in reinterpreting the derivative as an expectation. This is now stated and justified rigorously.

LEMMA 2.9. *Assume* **(R)**, **(E)** *and* **(H)**. *One has:*

$$\dot J(\alpha) = \int_0^T \mathbb{E}\Big(\sum_{i=1}^d \dot b_{i,t}\partial_{x_i}u(t,X_t) + \frac{1}{2}\sum_{i,j=1}^d [\dot{\sigma\sigma^*}]_{i,j,t}\partial^2_{x_i,x_j}u(t,X_t)\Big)dt.$$

Proof. The technical difficulty in the next computations comes from the possible explosion of derivatives of u for t close to T, when f is non smooth. For this reason, we first prove useful uniform estimates, which are standard: for any multi-index $\bar\gamma$, any smooth random variable $G \in \mathbb{D}^\infty$, any smooth function g with bounded derivatives and any parameters α and α', one has

$$\sup_{t\in[0,T[} |\mathbb{E}[G\ \partial_x^{\bar\gamma} u(t, X_t^{\alpha'})]| \le K(T,x)\frac{\|f\|_\infty}{T^{\frac{|\bar\gamma|}{2}}}\|G\|_{|\bar\gamma|,p'}. \tag{2.6}$$

Indeed, for $t \ge T/2$, first apply Proposition 2.4: then, use $|u(t,x)| \le \|f\|_\infty$ combined with some specific estimates for $\|H_{\bar\gamma}(X_t^{\alpha'}, G)\|_{\mathbf{L}^p} \le \frac{K(T,x)}{t^{\frac{|\bar\gamma|}{2}}}\|G\|_{|\bar\gamma|,p'}$ available under the ellipticity condition **(E)** (see Theorem 1.20 and Corollary 3.7 in Kusuoka and Stroock [KS84], or Section 4.1. in [Gob00] for a brief review). For $t \le T/2$, remark that using Markov property, one has $\partial_x^{\bar\gamma} u(t,x) = \partial_x^{\bar\gamma}\mathbb{E}\left(u(\frac{T+t}{2}, X^{t,x}_{\frac{T+t}{2}})\right) = \sum_{1\le|\gamma'|\le|\bar\gamma|}\mathbb{E}\left(\partial_x^{\gamma'} u(\frac{T+t}{2}, X^{t,x}_{\frac{T+t}{2}})G^{\gamma'}_{\frac{T+t}{2}}\right)$ with some clear notation; applying again the integration by part formula with the elliptic estimates gives $|\partial_x^{\bar\gamma} u(t,x)| \le \frac{K(T,x)}{[\frac{T+t}{2}-t]^{\frac{|\bar\gamma|}{2}}}\|f\|_\infty$ and (2.6) follows since $\frac{T+t}{2} - t \ge \frac{T}{4}$. Now, for $\epsilon \in \mathbb{R}$, the difference $J(\alpha+\epsilon) - J(\alpha)$ is equal to

$$\begin{aligned}
&\mathbb{E}\left(f(X_T^{\alpha+\epsilon}) - f(X_T^{\alpha})\right) = \mathbb{E}\left(u(T, X_T^{\alpha+\epsilon}) - u(0, X_0^{\alpha+\epsilon})\right)\\
&= \int_0^T \mathbb{E}\Bigg(\partial_t u(t, X_t^{\alpha+\epsilon}) + \sum_{i=1}^d b_i(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon)\partial_{x_i} u(t, X_t^{\alpha+\epsilon})\\
&\qquad + \frac{1}{2}\sum_{i,j=1}^d [\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon)\partial^2_{x_i,x_j} u(t, X_t^{\alpha+\epsilon})\Bigg)\, dt\\
&= \int_0^T \mathbb{E}\Bigg(\sum_{i=1}^d (b_i(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon) - b_i(t, X_t^{\alpha+\epsilon}, \alpha))\partial_{x_i} u(t, X_t^{\alpha+\epsilon})\\
&\qquad + \frac{1}{2}\sum_{i,j=1}^d ([\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon) - [\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha))\partial^2_{x_i,x_j} u(t, X_t^{\alpha+\epsilon})\Bigg)\, dt
\end{aligned}$$

where we used at the last equality the PDE solved by u to remove the term $\partial_t u$. Now, divide by ϵ and take its limit to 0: the result follows owing the uniform estimates (2.6). □

Remark that the formulation of Lemma 2.9 is strongly related to a form of the stochastic maximum principle (Pontryagin principle) for optimal control problems: the processes $([\partial_{x_i} u(t, X_t)]_i)_{0\le t<T}$ and $([\partial^2_{x_i,x_j} u(t, X_t)]_{i,j})_{0\le t<T}$ are the so-called adjoint processes (see Bensoussan [Ben88] for convex control domains, or more generally Peng [Pen90]) and solve Backward SDEs. Usually in these problems, the function f is taken to be smooth: here, since the law of X_t has a smooth density w.r.t. the Lebesgue measure, we can remove the regularity condition on f.

Note also that Lemma 2.9 remains valid under a hypoellipticity hypothesis (condition a) in Proposition 2.7). However, the derivation of tractable formulae below relies strongly on the ellipticity property.

2.3.2. Transformation using Itô-Malliavin integration by part formulae. The objective of this paragraph is to transform the expression for $\dot J(\alpha)$ in terms of explicit quantities. To remove the non-explicit terms $\partial_{x_i} u$ and $\partial^2_{x_i,x_j} u$, we may use some integration by part formulae, but here, to keep more tractable expressions, we are going to derive Bismut type formulae, i.e involving only Itô integrals instead of Skorohod integrals (see Bismut [Bis84]; Elworthy, Le Jan and Li [EJL99] and references therein), using a martingale argument (see also Thalmaier [Tha97] or more recently Picard [Pic02]). In the cited references, this approach has been used to compute some estimates of the gradient of u: here, we extend it to deal with higher derivatives. The basic tool is given by the following lemma.

LEMMA 2.10. *Assume* **(R)**, **(E)** *and define* $M_t = u'(t,X_t)Y_t$ *for* $t<T$. *Then* $M=(M_t)_{0\le t<T}$ *is a* $\mathbb{R}^1\otimes\mathbb{R}^d$*-valued martingale.*

Proof. The Markov property ensures that $(u(t,X_t^{0,x}))_{0\le t<T}$ is a martingale for any $x\in\mathbb{R}^d$. Hence, its derivative w.r.t. x (i.e. $(M_t)_{0\le t<T}$) is also a martingale (see Arnaudon and Thalmaier [AT98]). □

We now state a theorem which, if it is combined with Lemma 2.9, leads to an alternative representation for $\dot J(\alpha)$.

THEOREM 2.11. *Assume* **(R)** *and* **(E)**.

Under **(H)**, *one has*

$$\int_0^T \mathbb{E}\big(\sum_{i=1}^d \dot b_{i,t}\partial_{x_i}u(t,X_t)\big)dt=\mathbb{E}(H_T^{b,Adj.}), \tag{2.7}$$

where $H_T^{b,Adj.}=f(X_T)\int_0^T dt\ \dot b_t\cdot\dfrac{Z_t^*}{T-t}\int_t^T[\sigma_s^{-1}\ Y_s]^*\ dW_s$ *belongs to* $\bigcap_{p\ge1}\mathbf{L}^p$.

Under **(H')**, *one has*

$$\int_0^T \mathbb{E}\big(\sum_{i,j=1}^d [\sigma\dot{\sigma}^*]_{i,j,t}\partial^2_{x_i,x_j}u(t,X_t)\big)dt=\mathbb{E}(H_T^{\sigma,Adj.}), \tag{2.8}$$

where

$$\begin{aligned}H_T^{\sigma,Adj.}=&\int_0^T dt\sum_{i,j=1}^d[\sigma\dot{\sigma}^*]_{i,j,t}[f(X_T)-f(X_t)]\Big(\frac{2e^j}{T-t}\cdot\big[Z_t^*\int_{\frac{T+t}{2}}^T[\sigma_s^{-1}Y_s]^*dW_s\big]\\ &\times\frac{2e^i}{T-t}\cdot\big[Z_t^*\int_t^{\frac{T+t}{2}}[\sigma_s^{-1}Y_s]^*dW_s\big]+\frac{2e^i}{T-t}\cdot\big\{\nabla_x\big[Z_t^*\int_t^{\frac{T+t}{2}}[\sigma_s^{-1}Y_s]^*dW_s\big]Z_te^j\big\}\Big)\end{aligned}$$

belongs to $\bigcap_{p<p_0}\mathbf{L}^p$.

Proof. **Equality** (2.7)**.** First, Itô's formula applied to $u(t,X_t)$ gives an explicit form to the predictable representation theorem

$$\forall\ 0\le t\le\tau\le T\qquad u(\tau,X_\tau)=u(t,X_t)+\int_t^\tau u'(s,X_s)\sigma_s dW_s. \tag{2.9}$$

Since $(u'(t,X_t)Y_t)_{0\le t<T}$ is a martingale (see Lemma 2.10), we obtain that

$$\begin{aligned}u'(t,X_t)Y_t&=\mathbb{E}\big(\frac{1}{T-t}\int_t^T u'(s,X_s)Y_s\ ds|\mathcal{F}_t\big)\\&=\mathbb{E}\big(\frac{1}{T-t}\big[\int_t^T u'(s,X_s)\sigma_s dW_s\big]\big[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\big]^*|\mathcal{F}_t\big)\\&=\mathbb{E}\big(\frac{f(X_T)-u(t,X_t)}{T-t}\big[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\big]^*|\mathcal{F}_t\big)\\&=\mathbb{E}\big(\frac{f(X_T)}{T-t}\big[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\big]^*|\mathcal{F}_t\big),\end{aligned} \tag{2.10}$$

where we used at the third equality, the equation (2.9) with $\tau = T$ and $u(T, X_T) = f(X_T)$. Now, the proof of (2.7) is straightforward.

Equality (2.8)**.** Remark that a slight modification of the preceding arguments (namely integrating over $[t, (T+t)/2]$ instead of $[t, T]$ and applying (2.9) with $\tau = (t+T)/2$) leads to $\partial_{x_i} u(t, X_t) = \mathbb{E}\big(u\big(\frac{T+t}{2}, X_{\frac{T+t}{2}}\big)\frac{2e^i}{T-t}.\big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big]|\mathcal{F}_t\big)$. Differentiating w.r.t. x on both sides and using (2.10) yields

$$
\begin{aligned}
(\partial_{x_i} u)'(t, X_t) Y_t &= \mathbb{E}\big(u'\big(\frac{T+t}{2}, X_{\frac{T+t}{2}}\big) Y_{\frac{T+t}{2}} \frac{2e^i}{T-t} \cdot \big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big]|\mathcal{F}_t\big) \\
&\quad + \mathbb{E}\big(u\big(\frac{T+t}{2}, X_{\frac{T+t}{2}}\big) \frac{2e^i}{T-t} \cdot \nabla_x\big\{\big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big]\big\}|\mathcal{F}_t\big) \\
= \mathbb{E}\big([f(X_T) - f(X_t)]&\frac{2}{T-t}\big[\int_{\frac{T+t}{2}}^T [\sigma_s^{-1} Y_s]^* dW_s\big]^* \frac{2e^i}{T-t} \cdot \big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big] \\
&\quad + [f(X_T) - f(X_t)] \frac{2e^i}{T-t} \cdot \nabla_x\big\{\big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big]\big\}|\mathcal{F}_t\big)
\end{aligned}
$$

(note that terms with $f(X_t)$ has no contribution in expectation). Rearranging this last expression leads to (2.8).

The $\mathbf{L}^p$-estimates can be justified using the generalized Minkowski inequality and standard estimates from the stochastic calculus:

$$
\|H_T^{b,Adj.}\|_{\mathbf{L}^p} \leq \int_0^T \frac{\|f\|_\infty}{T-t} \|\dot{b}(t, X_t) \cdot Z_t^* \int_t^T [\sigma_s^{-1} Y_s]^* dW_s\|_{\mathbf{L}^p} dt \leq K(T, x) \int_0^T \frac{\|f\|_\infty}{\sqrt{T-t}} dt,
$$

$$
\|H_T^{\sigma,Adj.}\|_{\mathbf{L}^p} \leq K(T, x) \int_0^T \frac{\|f(X_T) - f(X_t)\|_{\mathbf{L}^{p'}}}{T-t}\, dt
$$

for $p < p' < p_0$. □

REMARK 2.1. *The terms $f(X_t)$ in $H_T^{\sigma,Adj.}$ seem to be crucial to ensure its $\mathbf{L}^p$ integrability: numerical experiments in Section 5 illustrate this fact.*

2.4. A third approach using martingales. We emphasize the dependence on α of the expected cost by denoting $u(\alpha, t, x) = \mathbb{E}(f(X_T^\alpha)|X_t^\alpha = x)$: hence, $J(\alpha) = u(\alpha, 0, X_0)$. From the estimates proved in Lemma 2.9, this is a differentiable function w.r.t. α and one has $|\dot{u}(\alpha, t, x)| \leq K(T, x)\|f\|_\infty$, as well $|u'(\alpha, t, x)| \leq \frac{K(T,x)}{\sqrt{T-t}}\|f\|_\infty$. The $\mathbf{L}_p$ estimates in the proof of Theorem 2.11 are more precise under **(H')**

$$
|\dot{u}(\alpha, t, x)| \leq K(T, x) \int_t^T \frac{\|f(X_T^{t,x}) - f(X_s^{t,x})\|_{\mathbf{L}^{p'}}}{T-s}\, ds
$$

for $p' < p_0$. As a consequence, if we put $g(r) = \mathbb{E}(\dot{u}(\alpha, r, X_r))$, we easily obtain $|g(r)| \leq K(T, x) \int_r^T \frac{\|f(X_T) - f(X_s)\|_{\mathbf{L}^{p_0}}}{T-s}\, ds$ and thus, $\lim_{r \to T} g(r) = 0$ under **(H')**.

For any $0 \leq r \leq s \leq T$, one has $\mathbb{E}(u(\alpha, r, X_r)) = \mathbb{E}(u(\alpha, s, X_s)) = \frac{1}{T-r}\int_r^T \mathbb{E}(u(\alpha, s, X_s))\, ds$ using the Markov property; hence, by differentiation w.r.t. α, one gets

$$
\begin{aligned}
\mathbb{E}\big(\dot{u}(\alpha, r, X_r)\big) &= \frac{1}{T-r} \int_r^T ds\, \mathbb{E}\big(\dot{u}(\alpha, s, X_s) + u'(\alpha, s, X_s)\dot{X}_s - u'(\alpha, r, X_r)\dot{X}_r\big) \\
&= \frac{1}{T-r} \int_r^T ds\, \mathbb{E}\big(\dot{u}(\alpha, s, X_s) + u'(\alpha, s, X_s)[\dot{X}_s - Y_s Z_r \dot{X}_r]\big)
\end{aligned}
$$

where we used at the last equality the martingale property of $M_t = u'(\alpha, t, X_t)Y_t$ between $t = s$ and $t = r$ (see Lemma 2.10).

Now, put $h(r) = \frac{1}{T-r}\int_r^T ds\, \mathbb{E}\Big(u'(\alpha, s, X_s)[\dot{X}_s - Y_s Z_r \dot{X}_r]\Big)$: one has derived the following integral equation

$$g(t) = \frac{1}{T-t}\int_t^T g(s)\, ds + h(t). \tag{2.11}$$

Before solving it, we express $h(r)$ using only f: for this, we use the predictable representation (2.9) which immediately gives

$$h(r) = \frac{1}{T-r}\mathbb{E}\big((f(X_T) - f(X_r))\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\big). \tag{2.12}$$

Note again that the term with $f(X_r)$ has no contribution and is put only to justify that $|h(r)| \leq K(T,x)\|f(X_T) - f(X_r)\|_{\mathbf{L}^{p_0}}$ (use the Burkholder-Davis-Gundy inequalities and straightforward upper bounds for $\|\dot{X}_s - Y_s Z_r \dot{X}_r\|_{\mathbf{L}^q} \leq K(T,x)\sqrt{s-r}$), from which we deduce that the integral $\int_0^T \frac{h(t)}{T-t}\, dt$ is convergent thanks to **(H')**. To solve the integral equation above, remark that $[\frac{1}{T-t}\int_t^T g(s)\, ds]' = -\frac{h(t)}{T-t}$, and thus $\frac{1}{T-t}\int_t^T g(s)\, ds = \mathrm{Cste} - \int_t^T \frac{h(r)}{T-r} dr$. The constant equals 0 since both integrals in the previous equality converge to 0 when t goes to T (use $\lim_{t\to T} g(t) = 0$ and **(H')**). Plug this new equality into (2.11), use (2.12) and take $t = 0$ (with $\dot{X}_0 = 0$) to get the following representation for $\dot{J}(\alpha)$: this is the main result of this section.

THEOREM 2.12. *Assume* **(R)**, **(E)** *and* **(H')**. *Then, one has* $\dot{J}(\alpha) = \mathbb{E}(H_T^{Mart.})$ *with*

$$\begin{aligned} H_T^{Mart.} &= \frac{f(X_T)}{T}\int_0^T [\sigma_s^{-1}\dot{X}_s]^* dW_s \\ &+ \int_0^T dr\, \frac{[f(X_T) - f(X_r)]}{(T-r)^2}\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s. \end{aligned} \tag{2.13}$$

Furthermore, the random variable $H_T^{Mart.}$ *belongs to* $\bigcap_{p<p_0}\mathbf{L}^p$.

Proof. What remains to prove is the $\mathbf{L}^p$ control of $H_T^{Mart.}$: this can be easily obtained combining Minkowski's inequality, Hölder's inequality, assumption **(H')**, and classical stochastic calculus inequalities as before. □

REMARK 2.2. When the parameter is not involved in the diffusion coefficient. *In that case, it is easy to see that the improved estimate* $\|\dot{X}_s - Y_s Z_r \dot{X}_r\|_{L_q} \leq K(T,x)$ $(s-r)$ *is available: thus, this enables to remove terms* $f(X_r)$ *in the expression of* $H_T^{Mart.}$, *without changing the finiteness of the* $\mathbf{L}^p$*-norm of the new* $H_T^{Mart.}$. *In other words, only assumption* **(H)** *is needed.*

Besides, still when α *is only in the drift coefficient, when these terms* $f(X_r)$ *are suppressed, it turns out that this representation coincides with that of Theorem 2.11. Indeed, let us transform* $P_r = \int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s = \int_r^T [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r \dot{X}_r]^* \int_r^T [\sigma_s^{-1} Y_s]^* dW_s := P_{1,r} - P_{2,r}$ *where*

$$\begin{aligned} P_{1,r} &= \int_0^T [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r \dot{X}_r]^* \int_0^T [\sigma_s^{-1} Y_s]^* dW_s \\ P_{2,r} &= \int_0^r [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r \dot{X}_r]^* \int_0^r [\sigma_s^{-1} Y_s]^* dW_s. \end{aligned}$$

From $Z_r\dot{X}_r = \int_0^r Z_s\dot{b}_s ds$ *(see* (1.6)*), one gets* $dP_{2,r} = [Z_r\dot{b}_r]^*(\int_0^r[\sigma_t^{-1}Y_t]^*dW_t)dr$, *hence is of bounded variation.* $P_{1,r}$ *is also of bounded variation, since* $Z_r\dot{X}_r$ *is. Thus, one obtains* $dP_r = -\dot{b}_r.Z_r^*(\int_r^T[\sigma_t^{-1}Y_t]^*dW_t)dr$*: furthermore, since* $P_T = 0$*, one has* $\|P_r\|_{\mathbf{L}^p} \leq K(T,x)(T-r)^{3/2}$*. An integration by parts formula in* (2.13) *now completes our assertion:*

$$H_T^{Mart.} = f(X_T)\left(\frac{1}{T}P_0 + \int_0^T \frac{P_r}{(T-r)^2}dr\right) = f(X_T)\left(-\int_0^T \frac{dP_r}{(T-r)}\right) = H_T^{b,Adj.}.$$

So, this martingale approach does not provide new elements when the parameter is not in the diffusion coefficient: on the contrary, if σ *depends on* α*, representations with the adjoint point of view or martingale one are really different (see numerical experiments). However, we must admit that this martingale approach remains to us a bit mysterious.*

3. Monte-Carlo simulation and analysis of the discretization error. In this section, we discuss the numerical implementation of the formulae derived in this paper to compute the sensitivity of $J(\alpha)$ w.r.t. α. These formulae are written as expectations of some functional of the process $(X_t)_{0\leq t\leq T}$ and related ones: a standard way to proceed consists in drawing independent simulations, approximating the functional using Euler schemes and averaging independent copies of the resulting functional to get an estimation of the expectation (see Section 5).

Here, we focus on the impact of the time step $h = T/N$ (N is the number of times discretization in the regular mesh of the interval $[0,T]$) in the simulation of the functional: it is well-known that for the evaluation of $\mathbb{E}(f(X_T))$, the discretization error using an Euler scheme is of order h (see Bally and Talay [BT96a] for measurable functions f, or Kohatsu-Higa and Pettersson [KHP02] if f is a distribution and for more general discretization schemes).

Here, the quantity of interest has a more complex structure, that is essentially $\mathbb{E}(f(X_T)H)$ where H is one of the random variable resulting from our computations. In general, H involves Itô or Skorohod integrals: our first objective is to give some approximation procedure to simulate these weights using only the increments of the Brownian motion computed along the regular mesh with time step h.

Our second objective is to analyze the error induced by this discretization procedure: generally speaking, the error is still at most linear w.r.t. h, as for the case $\mathbb{E}(f(X_T))$. The proofs are quite intricate and we postpone them to section 4.

Approximation procedure. We consider a regular mesh of the interval $[0,T]$, with N discretization times $t_i = ih$ where $h = T/N$ is the time step. Denote $\phi(t) = \sup\{t_i : t_i \leq t\}$. The processes we need to simulate are essentially $(X_t)_{0\leq t\leq T}$, $(Y_t)_{0\leq t\leq T}$, $(Z_t)_{0\leq t\leq T}$, $(\dot{X}_t)_{0\leq t\leq T}$ and we approximate them using a standard Euler scheme as follows:

$$X_t^N = x + \int_0^t b(\phi(s), X_{\phi(s)}^N)\,ds + \sum_{j=1}^q \int_0^t \sigma_j(\phi(s), X_{\phi(s)}^N)\,dW_s^j, \tag{3.1}$$

$$Y_t^N = \mathrm{I}_d + \int_0^t b'(\phi(s), X_{\phi(s)}^N)\,Y_{\phi(s)}^N\,ds + \sum_{j=1}^q \int_0^t \sigma_j'(\phi(s), X_{\phi(s)}^N)\,Y_{\phi(s)}^N\,dW_s^j, \tag{3.2}$$

$$Z_t^N = \mathrm{I}_d - \int_0^t Z_{\phi(s)}^N(b' - \sum_{j=1}^q(\sigma_j')^2)(\phi(s), X_{\phi(s)}^N)\,ds - \sum_{j=1}^q \int_0^t Z_{\phi(s)}^N\sigma_j'(\phi(s), X_{\phi(s)}^N)\,dW_s^j, \tag{3.3}$$

$$\dot{X}_t^N = \int_0^t \left(\dot{b}(\phi(s), X_{\phi(s)}^N) + b'(\phi(s), X_{\phi(s)}^N)\ \dot{X}_{\phi(s)}^N\right)\ ds$$
$$+\sum_{j=1}^q \int_0^t \left(\dot{\sigma}_j(\phi(s), X_{\phi(s)}^N) + \sigma_j'(\phi(s), X_{\phi(s)}^N)\ \dot{X}_{\phi(s)}^N\right)\ dW_s^j. \quad (3.4)$$

Note that only the increments $(W_{t_{i+1}}^j - W_{t_i}^j; 1 \leq j \leq q)_{0\leq i\leq N-1}$ of the Brownian motion are needed to get values of X^N, Z^N, Y^N, $\dot{X}^N$ at times $(t_i)_{0\leq i\leq N}$.

3.1. Path-wise approach. THEOREM 3.1. *Assume* **(R)**. *Then, one has*

$$\left|\dot{J}(\alpha) - \mathbb{E}\left(f'(X_T^N)\dot{X}_T^N\right)\right| \leq C(T,x,f)h,$$

under one of the two following assumptions on f and X:

A1) either f is of class C_b^4: in that case, one may put $C(T,x,f) = K(T,x)\sum_{1\leq|\alpha|\leq 4}\|\partial^\alpha f\|_\infty$.

A2) either f is continuously differentiable with a bounded gradient, under the non degeneracy condition **(E')**: *in that case, $C(T,x,f)$ can be taken equal to $K(T,x)\|f'\|_\infty\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q$ for some positive numbers p and q.*

Remark that in the case A1), only 3 additional derivatives of the function $g = f'$ are required to get the order 1 w.r.t. h: this is a slight improvement compared to results in Talay and Tubaro [TL90] where four derivatives are needed.

3.2. Malliavin calculus approach.

3.2.1. Elliptic case. One needs to define the approximation for the random variable $H_T^{Mall.Ell.} := \delta\big([\sigma^{-1}(.,X_.)\ Y_.\ Z_T\ \dot{X}_T]^*\big)$ involved in Proposition 2.5. Easy algebra using the equality (2.2) gives

$$H_T^{Mall.Ell.} = \sum_{i=1}^d \delta\big([\sigma^{-1}(.,X_.)\ Y_.]_i^*\ [Z_T\ \dot{X}_T]_i\big)$$
$$= \sum_{i=1}^d [Z_T\ \dot{X}_T]_i \int_0^T [\sigma^{-1}(s,X_s)\ Y_s]_i^*\ dW_s - \sum_{i=1}^d \int_0^T \mathcal{D}_s([Z_T\ \dot{X}_T]_i)[\sigma^{-1}(s,X_s)\ Y_s]_i\ ds.$$

The new quantities involved are $\mathcal{D}_s Z_{j,k,T}$ and $\mathcal{D}_s\dot{X}_{k,T}$: we now indicate how to simulate them. The $\mathbb{R}^{2d}$-valued process $\begin{pmatrix} X_t \\ \dot{X}_t \end{pmatrix}_{t\geq 0}$ forms a new stochastic differential equation (see equation (1.5)): we denote the flow of this extended system by $\hat{Y}_t$ and its inverse by $\hat{Z}_t$. As for Y_t and Z_t, we can define their Euler scheme (as in (3.2) and (3.3)), which we denote $\hat{Y}_t^N$ and $\hat{Z}_t^N$. The Malliavin derivative of this system follows from equation (2.3): hence, one has

$$\mathcal{D}_s\dot{X}_T = \Pi_d^{Row}\left(\hat{Y}_T\hat{Z}_s\begin{pmatrix} \vdots & \sigma_j(s,X_s) & \vdots \\ \vdots & \dot{\sigma}_j(s,X_s) + \sigma_j'(s,X_s)\ \dot{X}_s & \vdots \end{pmatrix}\right), \quad (3.5)$$

and we naturally approximate it by

$$[\mathcal{D}_s\dot{X}_T]^N = \Pi_d^{Row}\left(\hat{Y}_T^N\hat{Z}_s^N\begin{pmatrix} \vdots & \sigma_j(s,X_s^N) & \vdots \\ \vdots & \dot{\sigma}_j(s,X_s^N) + \sigma_j'(s,X_s^N)\ \dot{X}_s^N & \vdots \end{pmatrix}\right). \quad (3.6)$$

The same approach can be developed for the c-th column of the transpose of Z_T, since $\begin{pmatrix} X_t \\ (Z_t{}^*)_c \end{pmatrix}_{t\geq 0}$ forms a new SDE (see equation (1.4)): the associated flow $\hat{Y}_t{}^c$ and its inverse $\hat{Z}_t{}^c$ enable to derive an simple expression for $\mathcal{D}_s[(Z_t{}^*)_c]$ analogously to (3.5) and (3.6). As a consequence, one gets that

$$\mathcal{D}_s([Z_T\,\dot{X}_T]_i) = \mathbf{1}_{s\leq T}\sum_j A_{\beta(j,i),T}B_{\beta(j,i),s}, \tag{3.7}$$

where the finite sum above involves $A_{\beta(j,i),T}$ and $B_{\beta(j,i),s}$, which are given by some appropriate coordinates of the processes $\hat{Y}_T$, $(\hat{Y}_T{}^c)_{1\leq c\leq d}$ on one hand, and $\hat{Z}_s$, $(\hat{Z}_s{}^c)_{1\leq c\leq d}$, $\sigma_j(s,X_s)$, $\dot{\sigma}_j(s,X_s)$, $\sigma'_j(s,X_s)$, $\dot{X}_s$, Z_s on the other hand: for sake of simplicity, we do not explicit furthermore this expression (we refer to a technical report [GM02] for full details). Finally, we propose to approximate $H_T^{Mall.Ell.}$ by

$$\begin{aligned} H_T^{Mall.Ell.,N} &= \sum_{i=1}^d [Z_T^N\,\dot{X}_T^N]_i \int_0^T [\sigma^{-1}(\phi(s),X^N_{\phi(s)})\,Y^N_{\phi(s)}]_i^*\,dW_s \\ &\quad -\sum_{i=1}^d\int_0^T \Big(\sum_j A^N_{\beta(j,i),T}B^N_{\beta(j,i),\phi(s)}\Big)[\sigma^{-1}(\phi(s),X^N_{\phi(s)})\,Y^N_{\phi(s)}]_i\,ds, \end{aligned}$$

which can be simulated only using the Brownian increments as before. We now state that the approximation above converges at order 1 w.r.t. the time step.

THEOREM 3.2. *Assume* **(R)**, **(E)** *and* **(H)**. *For some $q\geq 0$, one has:*

$$\left|\dot{J}(\alpha) - \mathbb{E}\left(f(X_T^N)H_T^{Mall.Ell.,N}\right)\right| \leq K(T,x)\frac{\|f\|_\infty}{T^q}h.$$

3.2.2. General non degenerate case. Denote by $0_{d_1,d_2}$, the $d_1\times d_2$ matrix with 0 for each element: easy algebra yields that $\dot{V}_T^*\gamma_T^{-1}\mathcal{D}_sV_T$ equals

$$\begin{aligned} &\dot{V}_T^*\gamma_T^{-1}\Pi_r^{Row}(Y_TZ_s\sigma(s,X_s)) = (0_{1,d-r}\dot{V}_T^*)\begin{pmatrix} 0_{d-r,d-r} & 0_{d-r,r} \\ 0_{r,d-r} & \gamma_T^{-1}\end{pmatrix} Y_TZ_s\sigma(s,X_s) \\ &= \sum_{i=1}^d F_i[(Z_s\sigma(s,X_s))^*]_i \end{aligned}$$

where $F_i = \left(Y_T^*\begin{pmatrix} 0_{d-r,d-r} & 0_{d-r,r} \\ 0_{r,d-r} & \gamma_T^{-1}\end{pmatrix}\begin{pmatrix} 0_{d-r,1} \\ \dot{V}_T\end{pmatrix}\right)_i = \sum_j U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)}$: the random variables $(U_{\kappa(i,j),T})_{i,j}$ can be expressed as a product of coordinates of Y_T and $\dot{V}_T$; as before, we do not enter into full details to keep simple formulae to manipulate and we refer to [GM02].

Hence, the random variable of interest in Proposition 2.8, i.e. $H_T^{NonDeg.}$, equals

$$\delta\left(\dot{V}_T^*\gamma_T^{-1}\mathcal{D}_.V_T\right) = \sum_{i=1}^d F_i\int_0^T [(Z_s\sigma(s,X_s))^*]_i^*dW_s - \sum_{i=1}^d\int_0^T \mathcal{D}_sF_i[(Z_s\sigma(s,X_s))^*]_i ds.$$

By the chain rule, the Malliavin derivative of F_i is related to that of $U_{\kappa(i,j),T}$ (that is coordinates of Y_T and $\dot{V}_T$) and that of $(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)}$: the latest one can be expressed in terms of γ_T^{-1} and $\mathcal{D}_s\gamma_T$ (see Lemma 2.1.6 p.89 in Nualart [Nua95]) and we obtain

$$H_T^{NonDeg.} = \sum_{i,j} U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)} \int_0^T [(Z_s\sigma(s,X_s))^*]_i^* dW_s \tag{3.8}$$

$$-\sum_{i,j}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)} \int_0^T \mathcal{D}_s U_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_i ds \tag{3.9}$$

$$+\sum_{i,j,k,l} U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),k}(\gamma_T^{-1})_{l,\gamma(i,j)} \int_0^T \mathcal{D}_s(\gamma_{k,l,T})[(Z_s\sigma(s,X_s))^*]_i ds. \tag{3.10}$$

Analogously to the elliptic case, the integrals above will be discretized. Furthermore, the random variables $U_{\kappa(i,j),T}$ may be approximated by $U^N_{\kappa(i,j),T}$, defined by the same product of coordinates of Y_T^N and $\dot{V}_T^N$ that the one defining $U_{\kappa(i,j),T}$. Its weak derivative can be computed as in (3.7): indeed, by the same arguments, one can proved that

$$\mathcal{D}_s U_{\kappa(i,j),T} = \mathbf{1}_{s\le T}\sum_k \hat{U}_{\kappa(i,j,k),T}\check{U}_{\beta(i,j,k),s} \tag{3.11}$$

where $(\hat{U}_{\kappa(i,j,k),T})_{i,j,k}$ (resp. $(\check{U}_{\beta(i,j,k),s})_{i,j,k}$) are appropriate linear (resp. vector) values at time T (resp. at time s) of some extended systems of SDEs. Then, the natural approximation is

$$[\mathcal{D}_s U_{\kappa(i,j),T}]^N = \mathbf{1}_{s\le T}\sum_k \hat{U}^N_{\kappa(i,j,k),T}\check{U}^N_{\beta(i,j,k),s}. \tag{3.12}$$

Actually, the new feature compared to the elliptic case concerns the Malliavin covariance matrix γ_T and its weak derivative. Even if $\gamma_T = \int_0^T \Pi_r^{Row}(Y_T Z_s\sigma(s,X_s))$ $[\Pi_r^{Row}(Y_T Z_s\sigma(s,X_s))]^* ds$ is *a.s.* invertible with an inverse in any $\mathbf{L}^p$, a naive approximation may not enjoy these invertibility properties: for this reason, we add a small perturbation as follows

$$\gamma_T^N = \int_0^T \Pi_r^{Row}(Y_T^N Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))[\Pi_r^{Row}(Y_T^N Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))]^* ds + \frac{T}{N}\mathrm{I}_d. \tag{3.13}$$

This allows the following result.

LEMMA 3.3. *Assume* **(R)** *and* **(E')**. *Then, for any* $p\ge 1$, *one has for some positive numbers* p_1 *and* q_1*:* $\|1/\det(\gamma_T^N)\|_{\mathbf{L}^p} \le K(T,x)\|1/\det(\gamma_T)\|^{q_1}_{\mathbf{L}^{p_1}}$.

Proof. It is easy to check that $\|\gamma_T^N - \gamma_T\|_{\mathbf{L}^p} \le K(T,x)\sqrt{h}$ (use Lemma 4.2 below). Moreover, the eigenvalues of γ_T^N are all greater than h, hence $\det(\gamma_T^N)\ge h^r$, and one deduces

$$\begin{aligned}
\mathbb{E}(\det(\gamma_T^N)^{-p}) &= \mathbb{E}\left(\det(\gamma_T^N)^{-p}\mathbf{1}_{\det(\gamma_T^N)\le\frac12\det(\gamma_T)}\right) + \mathbb{E}\left(\det(\gamma_T^N)^{-p}\mathbf{1}_{\det(\gamma_T^N)>\frac12\det(\gamma_T)}\right)\\
&\le h^{-rp}\mathbb{P}\left(\frac{\det(\gamma_T)-\det(\gamma_T^N)}{\det(\gamma_T)}\ge\frac12\right) + 2^p\mathbb{E}(\det(\gamma_T)^{-p})\\
&\le h^{-rp}2^q\|\,|\det(\gamma_T)-\det(\gamma_T^N)|^q\|_{\mathbf{L}^{p_1}}\|\det(\gamma_T)^{-q}\|_{\mathbf{L}^{p_2}} + 2^p\mathbb{E}(\det(\gamma_T)^{-p})
\end{aligned}$$

where p_1 and p_2 are conjugate numbers. Take $q=2rp$ to get the result. □

To deal with the weak derivative of γ_T, one needs to rewrite

$$\gamma_{k,l,T} = \sum_{i'} A_{\epsilon(k,l,i'),T}\int_0^T B_{\eta(k,l,i'),u}du,$$

where $A_{\epsilon(k,l,i'),T}$ (resp. $B_{\eta(k,l,i'),u}$) are products of coordinates of Y_T (resp. Z_u and $\sigma(u,X_u)$). As for (3.7), the Malliavin derivative of $A_{\epsilon(k,l,i'),T}$ and $B_{\eta(k,l,i'),u}$ can be expressed as

$$\mathcal{D}_s A_{\epsilon(k,l,i'),T} = \mathbf{1}_{s\leq T} \sum_{j'} C_{\epsilon(k,l,i',j'),T} D_{\epsilon(k,l,i',j'),s}$$
$$\mathcal{D}_s B_{\eta(k,l,i'),u} = \mathbf{1}_{s\leq u} \sum_{j'} E_{\eta(k,l,i',j'),u} F_{\eta(k,l,i',j'),s}.$$

Hence, for $s \leq T$, one has

$$\begin{aligned}\mathcal{D}_s \gamma_{k,l,T} = &\sum_{i',j'} C_{\epsilon(k,l,i',j'),T} \left(\int_0^T B_{\eta(k,l,i'),u} du\right) D_{\epsilon(k,l,i',j'),s} \\ &+ \sum_{i',j'} A_{\epsilon(k,l,i'),T} F_{\eta(k,l,i',j'),s} \int_s^T E_{\eta(k,l,i',j'),u} du, \end{aligned} \tag{3.14}$$

which can be approximated by

$$\begin{aligned}[\mathcal{D}_s \gamma_{k,l,T}]^N = &\sum_{i',j'} C^N_{\epsilon(k,l,i',j'),T} \left(\int_0^T B^N_{\eta(k,l,i'),\phi(u)} du\right) D^N_{\epsilon(k,l,i',j'),s} \\ &+ \sum_{i',j'} A^N_{\epsilon(k,l,i'),T}\, F^N_{\eta(k,l,i',j'),s} \int_s^T E^N_{\eta(k,l,i',j'),\phi(u)} du. \end{aligned} \tag{3.15}$$

We now turn to the global approximation of the weight $H_T^{NonDeg.}$:

$$H_T^{NonDeg.,N} = \sum_{i,j} U^N_{\kappa(i,j),T} [(\gamma_T^N)^{-1}]_{\beta(i,j),\gamma(i,j)} \int_0^T [(Z^N_{\phi(s)} \sigma(\phi(s), X^N_{\phi(s)}))^*]_i^* dW_s \tag{3.16}$$
$$- \sum_{i,j} [(\gamma_T^N)^{-1}]_{\beta(i,j),\gamma(i,j)} \int_0^T [\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N [(Z^N_{\phi(s)} \sigma(\phi(s), X^N_{\phi(s)}))^*]_i ds \tag{3.17}$$
$$\begin{aligned}&+ \sum_{i,j,k,l} U^N_{\kappa(i,j),T} [(\gamma_T^N)^{-1}]_{\beta(i,j),k} [(\gamma_T^N)^{-1}]_{l,\gamma(i,j)} \\ &\qquad \int_0^T [\mathcal{D}_{\phi(s)}(\gamma_{k,l,T})]^N [(Z^N_{\phi(s)} \sigma(\phi(s), X^N_{\phi(s)}))^*]_i ds. \end{aligned} \tag{3.18}$$

We are now in position to state the following approximation result.

THEOREM 3.4. *Assume* **(R)**, **(E')** *and* **(H)**. *For some positive numbers p and q, one has:*

$$\left|\dot{J}(\alpha) - \mathbb{E}(f(V_T^N) H_T^{NonDeg.,N})\right| \leq K(T,x) \|f\|_\infty \|1/\det(\gamma_T)\|^q_{\mathbf{L}^p} h.$$

In the hypoelliptic case (Case a) in Proposition 2.7), note that the weak approximation result above holds true under a non degeneracy condition stated only at the initial point $(0, X_0)$ and not in the whole space as in [BT96a]: which is a significant improvement.

3.3. Adjoint approach. To approximate $H_T^{b,Adj.}$ and $H_T^{\sigma,Adj.}$ from Theorem 2.11, we propose the following natural estimates:

$$H_T^{b,Adj.,N} = f(X_T^N)h\sum_{k=0}^{N-1} \dot{b}(t_k, X_{t_k}^N) \cdot \frac{Z_{t_k}^{N*}}{T-t_k}\int_{t_k}^T [\sigma^{-1}(\phi(s), X_{\phi(s)}^N)\, Y_{\phi(s)}^N]^*\, dW_s, \quad (3.19)$$

$$\begin{aligned} H_T^{\sigma,Adj.,N} &= h\sum_{k=0}^{N-1}\sum_{i,j=1}^{d} [\sigma\dot{\sigma}^*]_{i,j}(t_k, X_{t_k}^N)[f(X_T^N) - f(X_{t_k}^N)] \\ &\quad\times\Big(\frac{2e^j}{T-t_k}\cdot\big[Z_{t_k}^{N*}\int_{\phi(\frac{T+t_k}{2})}^{T}[\sigma^{-1}(\phi(s), X_{\phi(s)}^N)Y_{\phi(s)}^N]^* dW_s\big] \\ &\quad\times\frac{2e^i}{T-t_k}\cdot\big[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s), X_{\phi(s)}^N)Y_{\phi(s)}^N]^* dW_s\big] \\ &\quad+\frac{2e^i}{T-t_k}\cdot\big\{\nabla_x\big[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s), X_{\phi(s)}^N)Y_{\phi(s)}^N]^* dW_s\big]Z_{t_k}^N e^j\big\}\Big). \end{aligned} \quad (3.20)$$

Derivatives $\nabla_x Y_{\phi(s)}^N$ and $\nabla_x Z_{t_k}^N$ are obtained by a direct differentiation in (3.2) and (3.3): we do not explicit the equations, which coincide with those of the Euler procedure applied to $\nabla_x Y_t$ and $\nabla_x Z_t$ defined in (1.3) and (1.4).

These approximations also induce a discretization error in the computation of $\dot{J}(\alpha)$ of order 1 w.r.t. h.

THEOREM 3.5. *Assume* **(R)**, **(E)** *and* **(H)**. *For some $p \geq 0$, one has:*

$$\Big|\dot{J}(\alpha) - \mathbb{E}\Big(H_T^{b,Adj.,N} + H_T^{\sigma,Adj.,N}\Big)\Big| \leq K(T,x)\frac{\|f\|_\infty}{T^p}h.$$

The proof is postponed to section 4.4.

3.3.1. Martingale approach. The natural approximation of $H_T^{Mart.}$ defined in Theorem 2.12 may be given by

$$\begin{aligned} H_T^{Mart.,N} = \frac{f(X_T^N)}{T}\int_0^T[\sigma^{-1}(\phi(s), X_{\phi(s)}^N)\dot{X}_{\phi(s)}^N]^* dW_s + \int_0^T dr\, \frac{[f(X_T^N) - f(X_{\phi(r)}^N)]}{(T-\phi(r))^2} \\ \times\int_{\phi(r)}^T[\sigma^{-1}(\phi(s), X_{\phi(s)}^N)(\dot{X}_{\phi(s)}^N - Y_{\phi(s)}^N Z_{\phi(r)}^N \dot{X}_{\phi(r)}^N)]^* dW_s. \end{aligned}$$

Unfortunately, we have not been able to analyze under the fairly general assumption **(H')**, the approximation error $\dot{J}(\alpha) - \mathbb{E}(H_T^{Mart.,N})$. Indeed, an immediate issue to handle would be to quantify the quality of the approximation of $\int_0^T dr\ \mathbb{E}\big(\frac{[f(X_T)-f(X_r)]}{(T-r)^2}$ $\int_r^T[\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\big)$ by its Riemann sum, which seems to be far from obvious under **(H')**.

4. Proof of the results on the discretization error analysis. This section is devoted to the proof of Theorems from Section 3 analyzing the discretization error.

The trick to prove these estimates for $\mathbb{E}(f(X_T))$ relies usually on the Markov property: one decomposes the error using the PDE solved by the function $(t,x) \mapsto \mathbb{E}(f(X_{T-t}^x))$ (see Bally and Talay [BT96a]), but this is meaningless in our situation. An other way to do consists in using cleverly the duality relationship (2.1) with some stochastic expansion to get the right order (see Kohatsu-Higa [KH01]). Here, we are going to mix and adapt the two mentioned approaches to get the expected results.

The first proof concerns the general non degenerate case: it is clearly the most intricate to obtain, other cases will be simpler to handle. Before this, we need to define some particular forms of stochastic expansions.

4.1. Preliminary results about some stochastic expansions. By convention, we set $dW_s^0 = ds$.

DEFINITION 4.1. *The real random variable U_T (which may depend on N) satisfies Property $(\mathcal{P})$ if it can be written as*

$$U_T = \sum_{i,j=0}^{q} c_{i,j}^{U,0}(T)\int_0^T c_{i,j}^{U,1}(t)\Big(\int_{\phi(t)}^t c_{i,j}^{U,2}(s)dW_s^i\Big)dW_t^j$$
$$+\sum_{i,j,k=0}^{q} c_{i,j,k}^{U,0}(T)\int_0^T c_{i,j,k}^{U,1}(t)\Big[\int_0^t c_{i,j,k}^{U,2}(s)\Big(\int_{\phi(s)}^s c_{i,j,k}^{U,3}(u)dW_u^i\Big)dW_s^j\Big]dW_t^k,$$

for some adapted processes $\{(c_{i,j}^{U,i_1}(t), c_{i,j,k}^{U,i_2}(t))_{t\geq 0} : 0\leq i,j,k\leq q, 0\leq i_1\leq 2, 0\leq i_2\leq 3\}$ (possibly depending on N). Moreover, the previous processes evaluated at a fixed time $t\in[0,T]$ belong to $\mathbb{D}^\infty$ and their Sobolev norms satisfy $\sup_N\sup_{t\in[0,T]}\Big(\|c_{i,j}^{U,i_1}(t)\|_{k',p}+\|c_{i,j,k}^{U,i_2}(t)\|_{k',p}\Big)<\infty$ for any $k',p\geq 1$.

We now give a simple but crucial lemma, which states that the error comparing a Brownian SDE to its Euler approximation fulfills Property $(\mathcal{P})$. For more general driven semimartingales, see Jacod and Protter [JP98].

LEMMA 4.2. *Consider a general d'-dimensional SDE $(\bar X_t)_{t\geq 0}$ defined by C^∞ coefficients with bounded derivatives, and $(\bar X_t^N)_{t\geq 0}$ its Euler approximation:*

$$\bar X_t = x+\int_0^t \bar b(s,\bar X_s)\,ds+\sum_{j=1}^q\int_0^t\bar\sigma_j(s,\bar X_s)\,dW_s^j,$$
$$\bar X_t^N = x+\int_0^t \bar b(\phi(s),\bar X_{\phi(s)}^N)\,ds+\sum_{j=1}^q\int_0^t\bar\sigma_j(\phi(s),\bar X_{\phi(s)}^N)\,dW_s^j.$$

Then, for each t, each component of $\bar X_t-\bar X_t^N$ satisfies $(\mathcal{P})$. Namely, for $1\leq k\leq d'$, one has

$$\bar X_{k,t}-\bar X_{k,t}^N=\sum_{i,j=0}^q c_{i,j,k}^{\bar X,0}(t)\int_0^t c_{i,j,k}^{\bar X,1}(s)\Big(\int_{\phi(s)}^s c_{i,j,k}^{\bar X,2}(u)dW_u^i\Big)dW_s^j$$

for some adapted processes $\{(c_{i,j,k}^{\bar X,i_1}(t))_{t\geq 0} : 0\leq i,j\leq q, 1\leq k\leq d', 0\leq i_1\leq 2\}$ satisfying $\sup_N\sup_{t\in[0,T]}\|c_{i,j,k}^{\bar X,i_1}(t)\|_{k',p}<\infty$ for any $k',p\geq 1$.

Proof. One has $\bar X_t-\bar X_t^N=\int_0^t\bar b'(s)(\bar X_s-\bar X_s^N)\,ds+\sum_{j=1}^q\int_0^t\bar\sigma_j'(s)(\bar X_s-\bar X_s^N)\,dW_s^j+\int_0^t[\bar b(s,\bar X_s^N)-\bar b(\phi(s),\bar X_{\phi(s)}^N)]\,ds+\sum_{j=1}^q\int_0^t[\bar\sigma_j(s,\bar X_s^N)-\bar\sigma_j(\phi(s),\bar X_{\phi(s)}^N)]\,dW_s^j$ with $a'(s)=\int_0^1\nabla_x a(s,\bar X_s^N+\lambda(\bar X_s-\bar X_s^N))d\lambda$ for $a=\bar b$ or $a=\bar\sigma_j$. Now, consider the unique solution of the linear equation $\mathcal{E}_t=\mathrm{I}_d+\int_0^t\bar b'(s)\mathcal{E}_s\,ds+\sum_{j=1}^q\int_0^t\bar\sigma_j'(s)\mathcal{E}_s\,dW_s^j$.

From Theorem 56 p.271 in Protter [Pro90], one deduces that

$$\begin{aligned}\bar{X}_t - \bar{X}_t^N = \mathcal{E}_t \int_0^t \mathcal{E}_s{}^{-1}\{[\bar{b}(s,\bar{X}_s^N) - \bar{b}(\phi(s),\bar{X}_{\phi(s)}^N)]\\ - \sum_{j=1}^q \bar{\sigma}_j'(s)[\bar{\sigma}_j(s,\bar{X}_s^N) - \bar{\sigma}_j(\phi(s),\bar{X}_{\phi(s)}^N)]\}\, ds\\ + \sum_{j=1}^q \mathcal{E}_t \int_0^t \mathcal{E}_s{}^{-1}[\bar{\sigma}_j(s,\bar{X}_s^N) - \bar{\sigma}_j(\phi(s),\bar{X}_{\phi(s)}^N)]\, dW_s^j;\end{aligned}$$

then, once applied Itô's formula between $\phi(s)$ and s, we can easily complete the proof of Lemma 4.2. □

4.2. Proof of Theorem 3.4 (general non degenerate case). To analyze the discretization error, for technical reasons, we may use a regularized function f and regularized random variables V_T and V_T^N. Let us briefly formalize this step.

We put $V_T^\epsilon = V_T + \epsilon\tilde{W}_T$ and $V_T^{N,\epsilon} = V_T^N + \epsilon\tilde{W}_T$, where $(\tilde{W}_T)_{t\geq 0}$ is an extra independent r-dimensional Brownian motion and we define $V_T^{\lambda,N,\epsilon} = V_T^{N,\epsilon} + \lambda(V_T^\epsilon - V_T^{N,\epsilon})$ for $\lambda \in [0,1]$. In the following computations, the Malliavin calculus will be made w.r.t. the $(q+r)$-dimensional Brownian motion $\begin{pmatrix} W_t \\ \tilde{W}_t \end{pmatrix}_{0\leq t\leq T}$.

Denote by $\bar{\mu}$ the measure defined by $\int_{\mathbb{R}^r} g(x)\bar{\mu}(dx) = \mathbb{E}(g(V_T^{0,N,0})) + \mathbb{E}(g(V_T^{1,N,0})) + \int_0^1 \mathbb{E}(g(V_T^{\lambda,N,0}))d\lambda$ and consider $(f_m)_{m\geq 1}$ a sequence of continuous functions with compact support, which converges to f in $\mathbf{L}^2(\bar{\mu})$. Thus, one easily gets that

$$\lim_{m\uparrow\infty}\lim_{\epsilon\downarrow 0} \|f_m^2(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2} = \lim_{m\uparrow\infty} \|f_m^2(V_T^{\lambda,N,0})\|_{\mathbf{L}^2} = \|f^2(V_T^{\lambda,N,0})\|_{\mathbf{L}^2} \leq \|f\|_\infty^2 \tag{4.1}$$

for $\lambda = 0$ or 1, and

$$\begin{aligned}\lim_{m\uparrow\infty}\lim_{\epsilon\downarrow 0}\left(\int_0^1 \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2} d\lambda\right) &= \lim_{m\uparrow\infty}\left(\int_0^1 \|f_m(V_T^{\lambda,N})\|_{\mathbf{L}^2} d\lambda\right)\\ &\leq \lim_{m\uparrow\infty}\sqrt{\int_0^1 \mathbb{E}(f_m^2(V_T^{\lambda,N}))d\lambda} = \sqrt{\int_0^1 \mathbb{E}(f^2(V_T^{\lambda,N,0}))d\lambda} \leq \|f\|_\infty. \end{aligned}\tag{4.2}$$

Then, the error to analyze is equal to $\dot{J}(\alpha) - \mathbb{E}(f(V_T^N)H_T^{NonDeg.,N}) = \lim_{m\uparrow\infty,\epsilon\downarrow 0}[\mathcal{E}_1(m,\epsilon) + \mathcal{E}_2(m,\epsilon)]$ with

$$\begin{aligned}\mathcal{E}_1(m,\epsilon) &= \mathbb{E}\left(f_m(V_T^\epsilon)H_T^{NonDeg.} - f_m(V_T^{N,\epsilon})H_T^{NonDeg.}\right),\\ \mathcal{E}_2(m,\epsilon) &= \mathbb{E}\left(f_m(V_T^{N,\epsilon})H_T^{NonDeg.} - f_m(V_T^{N,\epsilon})H_T^{NonDeg.,N}\right).\end{aligned}$$

In view of (4.1) and (4.2), it is enough to prove the following estimates, with some constants $K(T,x)$, p and q uniform in m and $\epsilon \leq 1$:

$$\begin{aligned}|\mathcal{E}_1(m,\epsilon)| \leq K(T,x)\big(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}\\ + \int_0^1 \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2} d\lambda\big)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h, \end{aligned}\tag{4.3}$$

$$|\mathcal{E}_2(m,\epsilon)| \leq K(T,x)\left(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}\right)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h. \tag{4.4}$$

Let $\psi \in C_b^\infty(\mathbb{R},\mathbb{R})$ be a cutting function verifying $\mathbf{1}_{[\frac{1}{2},\infty[} \le \psi \le \mathbf{1}_{[\frac{1}{4},\infty[}$: put

$$\psi_T^{N,\epsilon} = \psi\left(\frac{\inf_{\lambda\in[0,1]}\det(\gamma^{V_T^{\lambda,N,\epsilon}})}{\det(\gamma^{V_T^\epsilon})}\right).$$

4.2.1. Error $\mathcal{E}_1(m,\epsilon)$. Using a Taylor's formula, one can write $\mathcal{E}_1(m,\epsilon) = \mathcal{E}_{1,1}(m,\epsilon) + \mathcal{E}_{1,2}(m,\epsilon)$ with

$$\mathcal{E}_{1,1}(m,\epsilon) = \mathbb{E}\left([f_m(V_T^\epsilon) - f_m(V_T^{N,\epsilon})](1-\psi_T^{N,\epsilon})H_T^{NonDeg.}\right), \tag{4.5}$$

$$\mathcal{E}_{1,2}(m,\epsilon) = \int_0^1 d\lambda\, \mathbb{E}\left(f'_m(V_T^{\lambda,N,\epsilon})(V_T^\epsilon - V_T^{N,\epsilon})\psi_T^{N,\epsilon}H_T^{NonDeg.}\right). \tag{4.6}$$

The first term can be easily bounded by $K(T,x)\frac{\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2}+\|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}}{T^{d'}}h^p$ for any $p\ge 1$: indeed, the neglecting contribution comes from the term $1-\psi_T^{N,\epsilon}$ which is different from 0, on the event $\{\frac{\det(\gamma^{V_T^\epsilon})-\inf_{\lambda\in[0,1]}\det(\gamma^{V_T^{\lambda,N,\epsilon}})}{\det(\gamma^{V_T^\epsilon})} \ge \frac{1}{2}\}$. But using techniques of Lemma 3.3 and noting that $\gamma^{V_T^\epsilon} = \gamma^{V_T} + \epsilon^2 \mathrm{I}_d$, it is easy to show that this event has a probability of order h^p for any $p \ge 1$.

Now, to deal with the term $\mathcal{E}_{1,2}(m,\epsilon)$, note that the difference $V_T^\epsilon - V_T^{N,\epsilon} = V_T - V_T^N$ can be expressed componentwise using Lemma 4.2 and it follows that $\mathcal{E}_{1,2}(m,\epsilon)$ can be split in a sum of terms

$$\begin{aligned}\mathcal{E}_{1,2,i,j,k}(m,\epsilon) = \int_0^1 & d\lambda \mathbb{E}\big(\partial_{x_k} f_m(V_T^{\lambda,N,\epsilon})\psi_T^{N,\epsilon}H_T^{NonDeg.}c_{i,j,k}^{\bar{V},0}(T) \\ & \int_0^T c_{i,j,k}^{\bar{V},1}(t)\big[\big(\int_{\phi(t)}^t c_{i,j,k}^{\bar{V},2}(s)dW_s^i\big)dW_t^j\big]\big),\end{aligned} \tag{4.7}$$

for $0 \le i,j \le q$. If i and j are different from 0, apply twice the duality relation (2.1) combined with Fubini's theorem to obtain that $\mathcal{E}_{1,2,i,j,k}(m,\epsilon)$ equals

$$\begin{aligned}&\int_0^1 d\lambda \int_0^T dt \mathbb{E}\big(\mathcal{D}_t^j\big[\partial_{x_k} f_m(V_T^{\lambda,N,\epsilon})\psi_T^{N,\epsilon}H_T^{NonDeg.}c_{i,j,k}^{\bar{V},0}(T)\big]c_{i,j,k}^{\bar{V},1}(t)\big(\int_{\phi(t)}^t c_{i,j,k}^{\bar{V},2}(s)dW_s^i\big)\big) \\ &= \int_0^1 d\lambda \int_0^T dt \int_{\phi(t)}^t ds \mathbb{E}\big(\mathcal{D}_s^i\{\mathcal{D}_t^j\big[\partial_{x_k} f_m(V_T^{\lambda,N,\epsilon})\psi_T^{N,\epsilon}H_T^{NonDeg.}c_{i,j,k}^{\bar{V},0}(T)\big]c_{i,j,k}^{\bar{V},1}(t)\}c_{i,j,k}^{\bar{V},2}(s)\big) \\ &= \sum_l \int_0^1 d\lambda \int_0^T dt \int_{\phi(t)}^t ds\ \mathbb{E}\big(\partial_x^{\gamma(l)} f_m(V_T^{\lambda,N,\epsilon})G_{s,t,T}^{l,\lambda,N}\big)\end{aligned}$$

where the length of the differentiation index $\gamma(l)$ is less than 3. If i and/or j equals 0, an analogous formula holds with $|\gamma(l)| \le 2$. The random variable $G_{s,t,T}^{l,\lambda,N}$ does not depend on ϵ, belongs to $\mathbb{D}^\infty$ with Sobolev norms uniformly bounded w.r.t. λ, N, s, t: furthermore, it is equal to 0 when $\psi_T^{N,\epsilon} = 0$, because of the local property of the derivative operator (see Proposition 1.3.7 p.44 in Nualart [Nua95]).

Since $\det(\gamma^{V_T^{\lambda,N,\epsilon}}) \ge \epsilon^{2r}$, one can apply Proposition 2.4, which yields $\mathbb{E}\big(\partial_x^{\gamma(l)} f_m(V_T^{\lambda,N,\epsilon})G_{s,t,T}^{l,\lambda,N}\big) = \mathbb{E}\big(f_m(V_T^{\lambda,N,\epsilon})H_{\gamma(l)}(V_T^{\lambda,N,\epsilon}, G_{s,t,T}^{l,\lambda,N})\big)$ for some iterated

Skorohod integral $H_{\gamma(l)}(V_T^{\lambda,N,\epsilon}, G_{s,t,T}^{l,\lambda,N})$. Due to the local property of the Skorohod integral (see Proposition 1.3.6 p.43 in Nualart [Nua95]), one has $H_{\gamma(l)}(V_T^{\lambda,N,\epsilon}, G_{s,t,T}^{l,\lambda,N}) = H_{\gamma(l)}(V_T^{\lambda,N,\epsilon}, G_{s,t,T}^{l,\lambda,N})\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}$ and applying the estimate from Proposition 2.4, one gets:

$$\|H_{\gamma(l)}(V_T^{\lambda,N,\epsilon}, G_{s,t,T}^{l,\lambda,N})\|_{\mathbf{L}^2} \leq C\|[\gamma^{V_T^{\lambda,N,\epsilon}}]^{-1}\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}\|_{\mathbf{L}^{p_1}}^{q_1}\|V_T^{\lambda,N,\epsilon}\|_{k_2,p_2}^{q_2}\|G_{s,t,T}^{l,\lambda,N}\|_{k_3,p_3},$$

for some integers $p_1, p_2, p_3, q_1, q_2, k_2, k_3$. It is easy to upper bound $\|V_T^{\lambda,N,\epsilon}\|_{k_2,p_2}$ and $\|G_{s,t,T}^{l,\lambda,N}\|_{k_3,p_3}$, uniformly in λ, N, s, t and $\epsilon \leq 1$. The estimation of $\|[\gamma^{V_T^{\lambda,N,\epsilon}}]^{-1}\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}\|_{\mathbf{L}^{p_1}}$ is straightforward to derive since on $\{\psi_T^{N,\epsilon} \neq 0\}$, $\det(\gamma^{V_T^{\lambda,N,\epsilon}}) \geq \frac{1}{4}\det(\gamma^{V_T})$ which has an inverse in any $\mathbf{L}^p$ (Lemma 3.3). One has proved that

$$|\mathcal{E}_{1,2,i,j,k}(m,\epsilon)| \leq K(T,x)\left(\int_0^1 \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2}d\lambda\right)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h;$$

this completes the estimation (4.3).

4.2.2. Error $\mathcal{E}_2(m,\epsilon)$. As before, this error can be split in two parts $\mathcal{E}_2(m,\epsilon) = \mathcal{E}_{2,1}(m,\epsilon) + \mathcal{E}_{2,2}(m,\epsilon)$ with

$$\mathcal{E}_{2,1}(m,\epsilon) = \mathbb{E}\big(f_m(V_T^{N,\epsilon})(1-\psi_T^{N,\epsilon})(H_T^{NonDeg.} - H_T^{NonDeg.,N})\big),$$
$$\mathcal{E}_{2,2}(m,\epsilon) = \mathbb{E}\big(f_m(V_T^{N,\epsilon})\psi_T^{N,\epsilon}(H_T^{NonDeg.} - H_T^{NonDeg.,N})\big).$$

As before, the contribution $\mathcal{E}_{2,1}(m,\epsilon)$ can be neglected since one can easily upper bound it by $K(T,x)(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2})\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h^{p'}$ for any $p' \geq 1$. To show that $\mathcal{E}_{2,2}(m,\epsilon)$ has the right order, let us assume for a moment that

$$H_T^{NonDeg.} - H_T^{NonDeg.,N} = h\,U_T + \sum_{i,j=0}^q c_{i,j}^{H,0}(T)\int_0^T c_{i,j}^{H,1}(t)\Big(\int_{\phi(t)}^t c_{i,j}^{H,2}(s)dW_s^i\Big)dW_t^j$$
$$+ \sum_{i,j,k=0}^q c_{i,j,k}^{H,0}(T)\int_0^T c_{i,j,k}^{H,1}(t)\Big[\int_0^t c_{i,j,k}^{H,2}(s)\Big(\int_{\phi(s)}^s c_{i,j,k}^{H,3}(u)dW_u^i\Big)dW_s^j\Big]dW_t^k, \tag{4.8}$$

with coefficients satisfying conditions from Property $(\mathcal{P})$ (namely $\sup_N \|U_T\|_{\mathbf{L}^p} + \sup_N \sup_{t\in[0,T]}(\|c_{i,j}^{H,i_1}(t)\|_{k',p} + \|c_{i,j,k}^{H,i_2}(t)\|_{k',p}) \leq K(T,x)\|1/\det(\gamma_T)\|_{\mathbf{L}^{p'(k',p)}}^{q'(k',p)}$ for any $k', p \geq 1$). Then, one gets that

$$\mathcal{E}_{2,2}(m,\epsilon) = h\mathbb{E}\big(f_m(V_T^{N,\epsilon})\psi_T^{N,\epsilon}U_T\big) \tag{4.9}$$
$$+ \sum_{i,j=0}^q \mathbb{E}\Big(f_m(V_T^{N,\epsilon})\psi_T^{N,\epsilon}c_{i,j}^{H,0}(T)\int_0^T c_{i,j}^{H,1}(t)\Big(\int_{\phi(t)}^t c_{i,j}^{H,2}(s)dW_s^i\Big)dW_t^j\Big) \tag{4.10}$$
$$+ \sum_{i,j=0}^q \mathbb{E}\Big(f_m(V_T^{N,\epsilon})\psi_T^{N,\epsilon}c_{i,j,k}^{H,0}(T)$$
$$\int_0^T c_{i,j,k}^{H,1}(t)\Big[\int_0^t c_{i,j,k}^{H,2}(s)\Big(\int_{\phi(s)}^s c_{i,j,k}^{H,3}(u)dW_u^i\Big)dW_s^j\Big]dW_t^k\Big). \tag{4.11}$$

The r.h.s. of (4.9) clearly satisfies the estimate given by the r.h.s. of (4.4). Terms (4.10) and (4.11) can be estimated using exactly the same techniques as for (4.7). Hence, if (4.8) holds, the inequality (4.4) follows and Theorem 3.4 is proved.

4.2.3. Proof of the decomposition (4.8)**.** The derivation of this representation relies heavily on Lemma 4.2: indeed, most of the approximated quantities result from an Euler scheme applied to some appropriate SDE. Thus, it is enough to separately look at each factor in $H_T^{NonDeg.}$ and $H_T^{NonDeg.,N}$, by proving that essentially, they are of the form $c_{i,j}^{U,0}(T)\int_0^T c_{i,j}^{U,1}(t)\left(\int_{\phi(t)}^t c_{i,j}^{U,2}(s)dW_s^i\right)dW_t^j$ or $c_{i,j,k}^{U,0}(T)\int_0^T c_{i,j,k}^{U,1}(t)\left[\int_0^t c_{i,j,k}^{U,2}(s)\left(\int_{\phi(s)}^s c_{i,j,k}^{U,3}(u)dW_u^i\right)dW_s^j\right]dW_t^k$, while the other factors just belong to $\mathbb{D}^\infty$ with uniformly bounded Sobolev norms. The origin of the term $h\ U_T$ in (4.8) is the additional term used in the perturbation (3.13) of the matrix γ_T.

a) The difference $U_{\kappa(i,j),T}-U^N_{\kappa(i,j),T}$ (involved in (3.8), (3.10), (3.16) and (3.18)) satisfies $(\mathcal{P})$, since $U_{\kappa(i,j),T}$ is the product of coordinates of Y_T and $\dot{V}_T$, so that we can conclude by a direct application of Lemma 4.2.

b) Using the expressions of γ_T and γ_T^N, one gets $\gamma_{k,l,T}-\gamma^N_{k,l,T}=-\delta_{k,l}h+\mathcal{E}_{3,1,k,l}+\mathcal{E}_{3,2,k,l}$ with

$$\begin{aligned}\mathcal{E}_{3,1,k,l}=&\int_0^T\left[\Pi_r^{Row}(Y_TZ_s\sigma(s,X_s))[\Pi_r^{Row}(Y_TZ_s\sigma(s,X_s))]^*\right]_{k,l}ds\\&-\int_0^T\left[\Pi_r^{Row}(Y_T^NZ_s^N\sigma(s,X_s^N))[\Pi_r^{Row}(Y_T^NZ_s^N\sigma(s,X_s^N))]^*\right]_{k,l}ds\\\mathcal{E}_{3,2,k,l}=&\int_0^T\left[\Pi_r^{Row}(Y_T^NZ_s^N\sigma(s,X_s^N))[\Pi_r^{Row}(Y_T^NZ_s^N\sigma(s,X_s^N))]^*\right]_{k,l}ds\\-\int_0^T&\left[\Pi_r^{Row}(Y_T^NZ^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))[\Pi_r^{Row}(Y_T^NZ^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))]^*\right]_{k,l}ds.\end{aligned}$$

Using Lemma 4.2 and the relation $a(s,X_s)-a(s,X_s^N)=a'(s)(X_s-X_s^N)$ with $a'(s)=\int_0^1\nabla_xa(s,X_s^N+\lambda(X_s-X_s^N))d\lambda$ available for smooth functions a, it is straightforward to see that $\mathcal{E}_{3,1,k,l}$ can be written as a sum of terms satisfying $(\mathcal{P})$. The same conclusion holds for $\mathcal{E}_{3,2,k,l}$ if we apply Itô's formula between $\phi(s)$ and s.
Finally, using that $1/\det(\gamma_T)$ and $1/\det(\gamma_T^N)$ belongs to any $\mathbf{L}^p$ ($p\geq 1$) owing Lemma 3.3, it follows that the difference $[\gamma_T^{-1}]_{k',l'}-[(\gamma_T^N)^{-1}]_{k',l'}$ (involved in (3.8), (3.9), (3.10), (3.16), (3.17) and (3.18)) can be expressed as the r.h.s. of (4.8).

c) Concerning (3.8) and (3.16), the difference $\int_0^T[(Z_s\sigma(s,X_s))^*]_i^*dW_s-\int_0^T[(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i^*dW_s$ is equal to a sum of two terms: $\int_0^T[(Z_s\sigma(s,X_s))^*]_i^*dW_s-\int_0^T[(Z_s^N\sigma(s,X_s^N))^*]_i^*dW_s$ and $\int_0^T[(Z_s^N\sigma(s,X_s^N))^*]_i^*dW_s-\int_0^T[(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i^*dW_s$. It is straightforward to check that the first contribution satisfies Property $(\mathcal{P})$ using Lemma 4.2, while for the second one, it follows from Itô's formula.

d) The approximation error between terms (3.9) and (3.17) also comes from the difference $\int_0^T\mathcal{D}_sU_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_ids-\int_0^T[\mathcal{D}_{\phi(s)}U_{\kappa(i,j),T}]^N[(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_ids:=\mathcal{E}_{4,1,i,j}+\mathcal{E}_{4,2,i,j}$ where

$$\mathcal{E}_{4,1,i,j}=\int_0^T\mathcal{D}_sU_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_ids-\int_0^T[\mathcal{D}_sU_{\kappa(i,j),T}]^N[(Z_s^N\sigma(s,X_s^N))^*]_ids$$

$$\mathcal{E}_{4,2,i,j} = \int_0^T [\mathcal{D}_s U_{\kappa(i,j),T}]^N [(Z_s^N \sigma(s, X_s^N))^*]_i ds - \int_0^T [\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N [(Z_{\phi(s)}^N \sigma(\phi(s), X_{\phi(s)}^N))^*]_i ds.$$

The error induced by the approximation between $Z_s\sigma(s,X_s)$, $Z_s^N\sigma(s,X_s^N)$ and $Z_{\phi(s)}^N\sigma(\phi(s),X_{\phi(s)}^N)$ can be handled as before using Lemma 4.2 and Itô's formula. To deal with $\mathcal{D}_s U_{\kappa(i,j),T}$, $[\mathcal{D}_s U_{\kappa(i,j),T}]^N$ and $[\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N$, we may remind their particular forms given by equations (3.11) and (3.12): it implies that Lemma 4.2 can be applied once again to the extended systems which help in defining $\mathcal{D}_s U_{\kappa(i,j),T}$ and provides a contribution error equal to a sum of terms satisfying Property $(\mathcal{P})$.

e) The difference $\int_0^T \mathcal{D}_s(\gamma_{k,l,T})[(Z_s\sigma(s,X_s))^*]_i ds - \int_0^T [\mathcal{D}_{\phi(s)}(\gamma_{k,l,T})]^N [(Z_{\phi(s)}^N\sigma(\phi(s),X_{\phi(s)}^N))^*]_i ds$ coming from (3.10) and (3.18) can be analyzed with the same arguments as before, if we take into account the specific form of the derivative $\mathcal{D}_s(\gamma_{k,l,T})$ and its approximation given by (3.14) and (3.15).

The proof of (4.8) is complete.

Remark 4.1. *As mentioned before, the idea of using a Malliavin integration by part formula to directly prove a weak approximation result is due to Kohatsu-Higa [KH01] (see also [KHP02]). However, we need to take additional warnings compared to the example of one-dimensional diffusion given in [KHP02], in particular because of the possible degeneracy of $\gamma^{V_T^{\lambda,N,0}}$. The question may be put as "is a convex combination of two positive matrices still a positive matrix?": in dimension 1, this is obviously true, and this is in general false in higher dimension, except if the two initial matrices are close to each other. This explains why we need to introduce the additional factor $\psi_T^{N,\epsilon}$.*

4.3. Proof of Theorems 3.1 (path-wise approach) and 3.2 (elliptic case).

4.3.1. Theorem 3.1 case A1). We give a simplified proof regarding the techniques developed for Theorem 3.4. Set $X_T^{\lambda,N} = X_T^N + \lambda(X_T - X_T^N)$ and let us write

$$\begin{aligned}
&\mathbb{E}\big(f'(X_T)\dot{X}_T - f'(X_T^N)\dot{X}_T^N\big)\\
&= \sum_{i=1}^d \mathbb{E}\Big(\int_0^1 d\lambda (\partial_{x_i} f)'(X_T^{\lambda,N})(X_T - X_T^N)\dot{X}_{i,T}\Big) + \sum_{i=1}^d \mathbb{E}\big(\partial_{x_i} f(X_T^N)(\dot{X}_{i,T} - \dot{X}_{i,T}^N)\big)\\
&= \sum_{i=1}^d \sum_{l:1\le|\gamma(l)|\le 4} \int_0^1 d\lambda \int_0^T dt \int_{\phi(t)}^t ds \mathbb{E}\big(\partial^{\gamma(l)} f(X_T^{\lambda,N}) G_{s,t,T}^{i,l,N}\big)\\
&\quad + \sum_{i=1}^d \sum_{l:1\le|\bar\gamma(l)|\le 3} \int_0^T dt \int_{\phi(t)}^t ds \mathbb{E}\big(\partial^{\bar\gamma(l)} f(X_T^N) \bar{G}_{s,t,T}^{i,l,N}\big)
\end{aligned} \tag{4.12}$$

where we have combined Fubini's theorem, equality (2.1) and Lemma 4.2 as we have done for $\mathcal{E}_{1,2,i,j,k}(m,\epsilon)$ in (4.7). It is straightforward to check that the random variables $G_{s,t,T}^{i,l,N}$ and $\bar{G}_{s,t,T}^{i,l,N}$ belong to $\mathbb{D}^\infty$, with $\|.\|_{k,p}$-norms bounded by $K(T,x)$: use in particular their $\mathbf{L}^p$ estimates to complete the proof of the statement A1).

4.3.2. Theorem 3.1 case A2). Essentially, we proceed as for the previous case A1), except once obtained (4.12), we integrate by part to get back error estimates depending only on $\|f'\|_\infty$. This is not possible to do directly like this, since $\det(\gamma^{X_T^{\lambda,N}})$ may not have an inverse in $\mathbf{L}^p$. Hence, as for Theorem 3.4, we have to add a small perturbation to the initial processes (i.e. $X_t^\epsilon = X_t + \epsilon \hat{W}_t$ and so on) and then introduce in the expectation to compute, the localizing factor $\psi_T^{N,\epsilon}$ which discards the *degeneracy* event. Other arguments remain unchanged and we omit further details.

4.3.3. Theorem 3.2. Apply the arguments given in the proof of Theorem 3.4, to finally obtain $\left|\dot{J}(\alpha) - \mathbb{E}(f(X_T^N)H_T^{Mall.Ell.,N})\right| \leq \frac{K(T,x)}{T}\|f\|_\infty \|1/\det(\gamma_T)\|_{\mathbf{L}_p}^q h$, for some positive numbers p and q; we omit the computation details. The estimate given in Theorem 3.2 immediately follows using the well-known upper bound

$$\|1/\det(\gamma_T)\|_{\mathbf{L}_p} \leq \frac{K(T,x)}{T^d} \tag{4.13}$$

(see Theorem 3.5 in Kusuoka and Stroock [KS84]).

4.4. Theorem 3.5 (adjoint approach). The first approximation easy to justify is the time discretization of the integral involved in Lemma 2.9. For this, we remark that the function

$$t \mapsto \mathbb{E}(\sum_{i=1}^d \dot{b}_i(t,X_t)\partial_{x_i}u(t,X_t) + \frac{1}{2}\sum_{i,j=1}^d [\sigma\dot{\sigma}^*]_{i,j}(t,X_t)\partial^2_{x_i,x_j}u(t,X_t))$$

is of class $C_b^1([0,T],\mathbb{R})$: indeed, it is a smooth function especially because u is, and the uniform controls of derivatives follow from estimates (2.6). Hence, it remains to prove the following upper bounds, uniformly in i,j:

$$\begin{aligned}&\big|\mathbb{E}(f(X_T)\dot{b}(t_k,X_{t_k})\cdot {Z_{t_k}}^* \int_{t_k}^T [\sigma^{-1}(s,X_s)\;Y_s]^*\;dW_s - f(X_T^N)\dot{b}(t_k,X_{t_k}^N)\cdot {Z_{t_k}^N}^*\\ &\quad\times \int_{t_k}^T [\sigma^{-1}(\phi(s),X^N_{\phi(s)})\;Y^N_{\phi(s)}]^*\;dW_s)\big| \leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)h,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\big|\mathbb{E}([\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k})f(X_T)e^j\cdot[{Z_{t_k}}^*\int_{\frac{T+t_k}{2}}^T[\sigma^{-1}(s,X_s)Y_s]^*dW_s]\\ &\quad\times e^i\cdot[{Z_{t_k}}^*\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(s,X_s)Y_s]^*dW_s] - [\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k}^N)f(X_T^N)\\ &\quad\times e^j\cdot[Z_{t_k}^{N*}\int_{\phi(\frac{T+t_k}{2})}^T[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s]\\ &\times e^i\cdot[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s])\big| \leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)^2h,\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\big|\mathbb{E}([\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k})f(X_T)e^i\cdot\{\nabla_x[{Z_{t_k}}^*\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(s,X_s)Y_s]^*dW_s]Z_{t_k}e^j\}\\ &-[\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k}^N)f(X_T^N)e^i\cdot\{\nabla_x[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s]Z_{t_k}^Ne^j\})\big|\\ &\leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)h.\end{aligned} \tag{4.16}$$

Note that terms with $f(X_{t_k})$ and $f(X^N_{t_k})$ have been removed since they do not contribute in the expectation. The three errors above can be analyzed with the same techniques; hence, we only give details for (4.15) which appears to be slightly more difficult to deal with. We may point out that the main additional effort compared to Theorem 3.4 consists in proving that the estimates above include factors $(T-t_k)$ or $(T-t_k)^2$. For convenience in the following proof, we make a slight simplification, assuming that the number of discretization times N is even, so that we have the equality $\phi(\frac{T+t_k}{2}) = \frac{T+t_k}{2}$. Otherwise, the additional approximation error can be easily estimated directly from the derivation of formulae of Theorem 2.11.

Before performing the analysis of terms above, we give an other version of estimates (2.6) applied to the Euler scheme:

$$\sup_{t\in[0,T[} |\mathbb{E}[G\ \partial_x^{\bar\gamma} u(t,X_t^N)]| \le K(T,x)\frac{\|f\|_\infty}{T^{\frac{|\bar\gamma|}{2}}}\|G\|_{|\bar\gamma|,p'}. \tag{4.17}$$

The proof is unchanged. Moreover, when Lemma 3.3 or Proposition 2.4 is used in this section, we will systematically take into account the *elliptic* estimate stated in (4.13).

We introduce appropriate notations for what is playing the role of the infinitesimal generator of the Euler scheme:

$$\mathcal{L}_{(s',z)}u(s,x) = \sum_{i=1}^d b_i(s',z)\partial_{x_i}u(s,x) + \frac{1}{2}\sum_{i,j=1}^d [\sigma\sigma^*(s',z)]_{i,j}\partial^2_{x_i,x_j}u(s,x).$$

The infinitesimal generator of the SDE is thus defined by $Lu(s,x) = \mathcal{L}_{(s,x)}u(s,x)$.

Let us denote by A the difference to estimate in (4.15). An application of Itô's formula to $f(.) = u(T,.)$ between times $\frac{T+t_k}{2}$ and T to both terms in A leads to, after some simplifications using the PDE solved by u, $A = A_1 - A_2$ with

$$\begin{aligned}
A_1 &= \mathbb{E}\Big(\int_{\frac{T+t_k}{2}}^T u'(s,X_s)Y_s ds\ a_1 - \int_{\frac{T+t_k}{2}}^T u'(s,X_s^N)Y^N_{\phi(s)}ds\ a_1^N\Big)\\
a_1 &= [\dot{\sigma\sigma^*}]_{i,j}(t_k,X_{t_k})\big(e^i\cdot[Z_{t_k}{}^*\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(s,X_s)Y_s]^*dW_s]\big)Z_{t_k}e^j\\
a_1^N &= [\dot{\sigma\sigma^*}]_{i,j}(t_k,X^N_{t_k})\big(e^i\cdot[Z^{N*}_{t_k}\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s]\big)Z^N_{t_k}e^j\\
A_2 &= \mathbb{E}([\dot{\sigma\sigma^*}]_{i,j}(t_k,X^N_{t_k})[\int_{\frac{T+t_k}{2}}^T(\mathcal{L}_{(\phi(s),X^N_{\phi(s)})}u - Lu)(s,X^N_s)ds]\\
&\qquad\times e^j\cdot[Z^{N*}_{t_k}\int_{\frac{T+t_k}{2}}^T[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s]\\
&\qquad\times e^i\cdot[Z^{N*}_{t_k}\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s]).
\end{aligned}$$

4.4.1. Term A_1. Let us write $A_1 = A_{11} + A_{12} + A_{13}$, with

$$\begin{aligned}
A_{11} &= \int_{\frac{T+t_k}{2}}^T \mathbb{E}\big([u'(s,X_s)Y_s - u'(s,X_s^N)Y_s^N]a_1^N\big)\,ds\\
A_{12} &= \int_{\frac{T+t_k}{2}}^T \mathbb{E}\Big([u'(s,X_s^N)Y_s^N - u'(s,X_s^N)Y^N_{\phi(s)}]a_1^N\Big)\,ds
\end{aligned}$$

$$A_{13} = \frac{(T-t_k)}{2}\mathbb{E}\left(u'(\frac{T+t_k}{2}, X_{\frac{T+t_k}{2}})Y_{\frac{T+t_k}{2}}(a_1 - a_1^N)\right)$$

where we have applied Fubini's theorem in A_{11}, A_{12} and taken into account in A_{13} that $(u'(s,X_s)Y_s)_{0\le s<T}$ is a martingale (see Lemma 2.10).

Term A_{11}**.** One has $A_{11} = A_{111} + A_{112}$, where $A_{111} = \int_{\frac{T+t_k}{2}}^{T-h^3}\mathbb{E}([u'(s,X_s)Y_s - u'(s,X_s^N)Y_s^N]a_1^N)ds$ and $A_{112} = \int_{T-h^3}^{T}\mathbb{E}\big([u'(s,X_s)Y_s - u'(s,X_s^N)Y_s^N]a_1^N\big)\,ds$. The second term A_{112} is immediately bounded, using (4.17), by $\frac{K(T,x)}{T^q}h^3 \le \frac{K(T,x)}{T^q}(T-t_k)^2h$.

We now handle A_{111}. For the technical reasons related to the integration by part formula as in the proof of Theorem 3.4, we modify X_t in $X_t^\epsilon = X_t + \epsilon\hat{W}_t$ and X_t^N in $X_t^{N,\epsilon} = X_t^N + \epsilon\hat{W}_t$ for an extra independent Brownian motion $\hat{W}$: however, at the end, we take the limit $\epsilon \downarrow 0$, owing the uniformity of the estimates below w.r.t. ϵ. Let us also introduce again the localizing factor $\psi_T^{N,\epsilon}$.

One writes $A_{111}^\epsilon = \int_{\frac{T+t_k}{2}}^{T-h^3}\mathbb{E}\Big([1-\psi_T^{N,\epsilon}][u'(s,X_s^\epsilon)Y_s - u'(s,X_s^{N,\epsilon})Y_s^N]a_1^N\Big)\,ds$ $+ \int_{\frac{T+t_k}{2}}^{T-h^3}\mathbb{E}\Big(\psi_T^{N,\epsilon}[u'(s,X_s^\epsilon)Y_s - u'(s,X_s^{N,\epsilon})Y_s^N]a_1^N\Big)\,ds$. Using a direct upper bound for $|u'(s,x)| \le \frac{K(T,x)}{\sqrt{T-s}} \le \frac{K(T,x)}{h^{3/2}}$ $(s \le T-h^3)$ and the fact that $\mathbb{P}(\psi_T^{N,\epsilon} \ne 1)$ is neglectible w.r.t. any power in h (see term (4.5)), the first contribution in A_{111}^ϵ is clearly bounded by $(T-t_k)K(T,x)h^p \le (T-t_k)^2K(T,x)h^{p-1}$ for any $p \ge 1$. The second term can be treated with the same techniques than for (4.6): furthermore, to transform the stochastic integral a_1^N into a Lebesgue integral (in order to get an additional factor $(T-T_k)$), we use once again the duality relation (2.1). Note also that the factor $1/s^q$ which usually appears in the upper bound estimate is well controlled since $s \ge T/2$: hence, one gets $|A_{111}^\epsilon| \le \frac{K(T,x)}{T^q}(T-T_k)^2h$ uniformly w.r.t. ϵ. This proves that $|A_{11}| \le \frac{K(T,x)}{T^q}(T-T_k)^2h$.

Term A_{12}**.** Apply Itô's formula to obtain that A_{12} is a sum of terms of the form $\int_{\frac{T+t_k}{2}}^{T}ds\int_{\phi(s)}^{s}dt\mathbb{E}\big(a_1^N\partial^{\bar\gamma}u(t,X_t^N)g_t^N\big)$ for some smooth random variables g_t^N such that $\sup_N\sup_{t\in[0,T]}\|g_t^N\|_{k,p} \le K(T,x)$. To transform the expression with a_1^N (as for A_{111}^ϵ), apply duality relation (2.1) and estimates (4.17) to obtain $|A_{12}| \le \frac{K(T,x)}{T^q}(T-T_k)^2h$ (here, we do not need to add a perturbation to X_t^N since $\gamma^{X_t^N}$ has already the required invertibility property under **(E)**).

Term A_{13}**.** An application of Lemma 4.2 ensures that we can decompose A_{13} in a sum of terms of the form

$$\frac{(T-t_k)}{2}\mathbb{E}\big(u'(\frac{T+t_k}{2}, X_{\frac{T+t_k}{2}})\big(\int_{t_k}^{\frac{T+t_k}{2}} c_l^{U,3}(t)dW_t^l\big) \\ \times c_{i,j}^{U,0}(T)\int_0^{t_k} c_{i,j}^{U,1}(t)\big(\int_{\phi(t)}^{t} c_{i,j}^{U,2}(s)dW_s^i\big)dW_t^j\big),$$

and

$$\frac{(T-t_k)}{2}\mathbb{E}\big(u'(\frac{T+t_k}{2}, X_{\frac{T+t_k}{2}})c_{i,j,l}^{U,0}(T) \\ \times\int_{t_k}^{\frac{T+t_k}{2}} c_{i,j,l}^{U,1}(t)\big[\int_0^t c_{i,j,l}^{U,2}(s)\big(\int_{\phi(s)}^{s} c_{i,j,l}^{U,3}(u)dW_u^i\big)dW_s^j\big]dW_t^l\big).$$

Apply techniques used for (4.7) (i.e. several times equality (2.1) combined with Fubini's theorem and finally estimates (4.17)) to conclude that $|A_{13}| \leq \frac{K(T,x)}{T^q}(T-T_k)^2h$.

The proof of $|A_1| \leq \frac{K(T,x)}{T^q}(T-T_k)^2h$ is complete.

4.4.2. Term A_2. Itô's formula gives $(\mathcal{L}_{(\phi(s),X^N_{\phi(s)})}u - Lu)(s, X^N_s) = \int_{\phi(s)}^s [\mathcal{A}u]_r dr + [\bar{A}u]_r dW_r$ with $[\mathcal{A}u]_r = (\partial_t + \mathcal{L}_{\phi(s),z})[\mathcal{L}_{\phi(s),z}u(t,x) - Lu(t,x)]|_{z=X^N_{\phi(s)},t=r,x=X^N_r}$ and $[\bar{A}u]_r = \nabla_x[\mathcal{L}_{\phi(s),z}u(t,x) - Lu(t,x)]|_{z=X^N_{\phi(s)},t=r,x=X^N_r}\sigma(\phi(s),X^N_{\phi(s)})$. Thus, A_2 can be split in two parts $A_2 = A_{21} + A_{22}$ with

$$A_{21} = \mathbb{E}\big(g_k \cdot [\int_{\frac{T+t_k}{2}}^T [\sigma^{-1}(\phi(s), X^N_{\phi(s)})Y^N_{\phi(s)}]^* dW_s] \int_{\frac{T+t_k}{2}}^T ds \int_{\phi(s)}^s [\mathcal{A}u]_r dr\big),$$

$$A_{22} = \mathbb{E}\big(g_k \cdot [\int_{\frac{T+t_k}{2}}^T [\sigma^{-1}(\phi(s), X^N_{\phi(s)})Y^N_{\phi(s)}]^* dW_s] \int_{\frac{T+t_k}{2}}^T ds \int_{\phi(s)}^s [\bar{A}u]_r dW_r\big),$$

$$g_k = [\sigma\dot{\sigma}^*]_{i,j}(t_k, X^N_{t_k})e^i \cdot [Z_{t_k}{}^* \int_{t_k}^{\frac{T+t_k}{2}} [\sigma^{-1}(\phi(s), X^N_{\phi(s)})Y^N_{\phi(s)}]^* dW_s] Z_{t_k} e^j.$$

Term A_{21}. Applying twice Fubini's theorem and duality relationship (2.1), one obtains that $A_{21} = \sum_{\bar{\gamma}:|\bar{\gamma}|\leq 5} \int_{\frac{T+t_k}{2}}^T ds \int_{\phi(s)}^s dr \int_{\frac{T+t_k}{2}}^T dt\ \mathbb{E}\big(\partial^{\bar{\gamma}} u(r, X^N_r) G^{\bar{\gamma},N}_{t,r,s}\big)$, for some smooth random variables $G^{\bar{\gamma},N}_{t,r,s}$ with $\|G^{\bar{\gamma},N}_{t,r,s}\|_{k,p} \leq K(T,x)$ uniformly in N,t,r,s. Thus, estimates (4.17) are sufficient to conclude that $|A_{21}| \leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)^2h$.

Term A_{21}. We first apply Fubini's theorem and the isometry property to get

$$A_{22} = \int_{\frac{T+t_k}{2}}^T ds \int_{\phi(s)}^s dr \mathbb{E}\big(g_k \cdot \int_{\frac{T+t_k}{2}}^T ([\bar{A}u]_r \sigma^{-1}(\phi(s), X^N_{\phi(s)})Y^N_{\phi(s)})^* dr\big),$$

and thus, by estimates (4.17), one obtains $|A_{22}| \leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)^2h$.

One has proved $|A_2| \leq K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)^2h$.

REMARK 4.2. *The powers of T involved in the upper bounds stated in Theorems 3.1, 3.2 and 3.5 have not been explicited. A more careful proof of estimates given in Proposition 2.4 adapted to the elliptic case (in the spirit of Kusuoka and Stroock [KS84]) would show that the exponent of T equals $\frac{1}{2}$. As a consequence, the analysis of the weak error for a terminal cost would extend to the instantaneous costs problems since $\int_0^T \frac{dt}{\sqrt{t}} < \infty$.*

5. Numerical experiments.

5.1. Analysis of computational complexity. In this paragraph we indicate the first order approximation of the number of elementary operations (multiplications) needed for computing the different estimators, w.r.t. the quantities m (number of parameters), d (dimension of the space), q (dimension of the Brownian motion) and N (number of time discretizations).

In previous sections we derived estimators of the gradient of the performance measure $J(\alpha)$ w.r.t. α when J is defined by a terminal cost (see (1.2)). However, these results may be extended to functionals with instantaneous costs such as $J(\alpha) = \mathbb{E}\left(\int_0^T g(t, X_t)dt + f(X_T)\right)$ for which an estimator of the gradient may be $\frac{T}{N}\sum_{i=1}^{N} H^N_{t_i}(g) + H^N_T(f)$ where $H^N_{t_i}(g)$ (respectively $H^N_T(f)$) is an approximated estimator of the gradient of $\mathbb{E}(g(t_i, X_{t_i}))$ (resp. $\mathbb{E}(f(X_T))$). This case is illustrated in the first numerical experiment considered below.

The computational complexity of the different estimators depends on whether the payoff has instantaneous costs (in which case an estimator $H^N_{t_i}(g)$ for all $i \in \{1, \cdots, N\}$ is needed) or if it has only a terminal cost (for which only $H^N_T(f)$ is needed). In the path-wise and Malliavin calculus methods, the cost of computing $H^N_{t_i}(g)$ for all $i \in \{1, \cdots, N\}$ is the same as just computing $H^N_T(f)$, whereas in the adjoint and martingale methods, there is an additional computational burden.

- Complexity of the path-wise method: $d^2 qmN$ operations for computing the path-wise estimator $H^{Path.,N}_{t_i}(g)$ (see Proposition 1.1) for all $i \in \{1, \cdots, N\}$ (required for computing $\dot{X}^N_{t_i}$, for all $i \in \{1, \cdots, N\}$ and all m parameters).
- Complexity of the Malliavin calculus method, in the elliptic case ($q = d$): $3d^4(d+m)N$ operations, for computing the Malliavin calculus estimator $H^{Mall.Ell.,N}_{t_i}(g)$ (see Proposition 2.5) for all $i \in \{1, \cdots, N\}$. Indeed, the complexity of computing the Malliavin derivative of each column c (among d) of Z^N_{c,t_i} is $3d^4N$, and computing the Malliavin derivative of $\dot{X}^N_{t_i}$ for all m parameters takes $3d^3mN$ operations.
- Complexity of the adjoint method: $d^4N^2 + d^2mN^2/2$ operations are needed to compute the adjoint estimator $H^{Adj.,N}_{t_i}(g) = H^{b,Adj.,N}_{t_i}(g) + \frac{1}{2}H^{\sigma,Adj.,N}_{t_i}(g)$ (see Lemma 2.9 and Theorem 2.11) for all $i \in \{1, \cdots, N\}$. Our implementation memorizes $Z^N_{t_i}$ (and other data) along the trajectory and computes $H^{b,Adj.,N}_{t_i}(g)$ and $H^{\sigma,Adj.,N}_{t_i}(g)$ for all $i \in \{1, \cdots, N\}$, afterwards. Such an implementation allows to treat problems with instantaneous costs. If we consider a problem with a terminal cost only, the complexity is reduced to $4d^4N + 3d^2mN$.
- Complexity of the martingale method: $d^2N^2/2 + dmN^2/2 + d^3mN$ for computing the martingale estimator $H^{Mart.,N}_{t_i}(g)$ (see Theorem 2.12) for all $i \in \{1, \cdots, N\}$. For problems with terminal cost only the complexity of computing $H^{Mart.,N}_T(f)$ is d^3mN.

These results are summarized in Table 5.1. Note that they are strongly related to the way we have implemented the methods and they are not guaranteed to be optimal.

TABLE 5.1
Complexity (in terms of number of elementary operations) of the different estimators for payoff with instantaneous costs or with terminal cost only.

	Path-wise	Malliavin	Adjoint	Martingale
Instantaneous costs	d^3mN	$3d^4(d+m)N$	$d^4N^2 + d^2m\frac{N^2}{2}$	$d(d+m)\frac{N^2}{2} + d^3mN$
Terminal cost	same	same	$4d^4N + 3d^2mN$	$d^2N + d^3mN$

5.2. Stochastic linear quadratic optimal control. We consider a simple one-dimensional Stochastic Linear Quadratic (SLQ) control problems (see [CY01] and [YZ99] for an extensive study on SLQ problems) for which the control $u(\cdot)$ appears in particular in the diffusion term: $dX_t = u(t)dt + \delta u(t)dW_t$. The cost functional to be minimized is $J(u(\cdot)) = \mathbb{E}\big[\int_0^1 X_t^2 dt\big]$. This problem admits an optimal control (see references above) given by the state feedback $u^*(t) = -\frac{X_t}{\delta^2}$.

We consider a class of feedback controllers $u(t, x, \alpha)$ linearly parameterized by a 3-

dimensional vector α with basis functions 1, x and t (i.e. $u(t,x,\alpha) = \alpha_1 + \alpha_2 x + \alpha_3 t$) and we write $J(\alpha)$ for $J(u(\cdot, X_\cdot, \alpha))$. In that case, the optimal control u^* belongs to the class of parameterized feedback controllers and corresponds to the parameter $\alpha^* = (0, -1/\delta^2, 0)$.

As explained before, since the payoff involves instantaneous costs, we evaluate $\nabla_\alpha J(\alpha)$ using a quantity of type $\frac{T}{N}\sum_{i=1}^N H_{t_i}^N(x^2)$. We check that the different estimators (Path-wise, Malliavin calculus, adjoint, martingale) return a zero gradient for the value α^* of the parameter and compare their variance and time for computation. Table 5.2 shows the empirical variance of the different estimators obtained for 1000 trajectories, with $h = 0.05$, $\delta = 1$. These simulations have been performed on a Pentium III, 700Mhz processor.

TABLE 5.2
Variance of the estimators $H^{Path.}$, $H^{Mall.Ell.}$, $H^{Adj.}$, and $H^{Mart.}$ of $\nabla_\alpha J(\alpha)$ at the optimal setting of the parameter: $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 0$.

Var(H)	Path-wise	Malliavin	Adjoint	Martingale
α_1	0.1346	0.3754	0.6669	0.1653
α_2	0.0525	0.1188	0.1707	0.0480
α_3	0.0136	0.0446	0.0612	0.0148
CPU Time	0.44s	1.95s	2.89s	0.89s

Notice that the estimator used in the adjoint approach includes the term $f(X_T) - f(X_t)$ in the computation of $H_T^{\sigma,Adj.}$. Table 5.3 shows similar results for a sub-optimal setting of the parameter (here α_1, α_2, and α_3 are chosen randomly within the range $[-0.1, 0.1]$). The columns Adjoint2 and Martingale2 describe simulations of the adjoint and martingale methods when the term $f(X_t)$ is omitted from the computation of the estimators $H_T^{\sigma,Adj.}$ and $H_T^{Mart.}$. We notice that the variance of these estimators are significantly larger than when the term $f(X_t)$ is included, which corroborates Remark 2.1.

TABLE 5.3
Variance of the different estimators of $\nabla_\alpha J(\alpha)$ for $\alpha_1 = -0.0789$, $\alpha_2 = 0.0156$, $\alpha_3 = 0.0648$.

Var(H)	Path-wise	Malliavin	Adjoint	Adjoint2	Mart.	Mart.2
α_1	0.2005	4.0347	1.0287	9.5535	1.5085	4.6029
α_2	0.0252	0.6597	0.1433	1.6781	0.2360	0.7894
α_3	0.0174	0.3869	0.1051	2.2337	0.1407	1.0185
CPU Time	0.44s	1.97s	2.94s	2.94s	0.90s	0.90s

For this problem with smooth cost functions, the path-wise approach provides the best performance in terms of variance of the estimator. This good behavior for smooth costs compared to other methods has been previously observed in [FLL+99].

5.2.1. Stochastic approximation algorithm. The computation of an estimator H of $\nabla_\alpha J(\alpha)$ may be used by a Stochastic Approximation algorithm (see e.g. [KY97] or [BMP90]) to search a locally optimal parameterization of the controller. The algorithm begins with an initial setting of the parameter α^0. Then, if α^k denotes the value of the parameter at iteration k, the algorithm proceeds by computing an estimator $\widehat{\nabla_\alpha J(\alpha^k)}$ of $\nabla_\alpha J(\alpha^k)$ and then by performing a stochastic gradient ascent

$$\alpha^{k+1} = \alpha^k + \eta_k \widehat{\nabla_\alpha J(\alpha^k)} \tag{5.1}$$

where the learning steps η_k satisfy a decreasing condition (for example $\sum_k \eta_k = \infty$ and $\sum_k \eta_k^2 < \infty$, see [Pol87]). Assuming smoothness conditions on $J(\alpha)$ and a bounded variance for $\widehat{\nabla_\alpha J(\alpha^k)}$, one proves that if α^k converges, then the limit is a point of local minimum for $J(\alpha)$ (see references above for several sets of hypothesis for which the convergence is guaranteed).

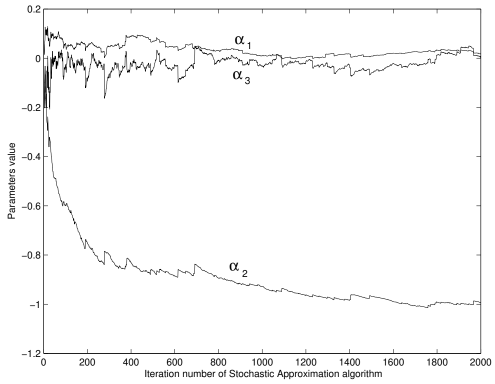


FIGURE 5.1. *Stochastic approximation of the control parameters. The gradient $\nabla_\alpha J(\alpha_k)$ is estimated using the path-wise method.*

Figure 5.1 illustrates this algorithm on the SLQ problem described previously, where the initial parameter is chosen randomly (same value as in Table 5.3). At iteration k, one trajectory is simulated using the controller parameterized by α^k and an estimation $\widehat{\nabla_\alpha J(\alpha^k)}$ of $\nabla J(\alpha_k)$ (using the path-wise method) is obtained. The parameter is updated according to (5.1) with a learning step $\eta_k = \frac{K}{K+k}$ (with $K = 200$). We notice that the parameter converges to $\alpha^* = (0, -1, 0)$.

The speed of convergence for such algorithms is closely related of the variance of the gradient estimator, which motivates our variance analysis for the different estimators.

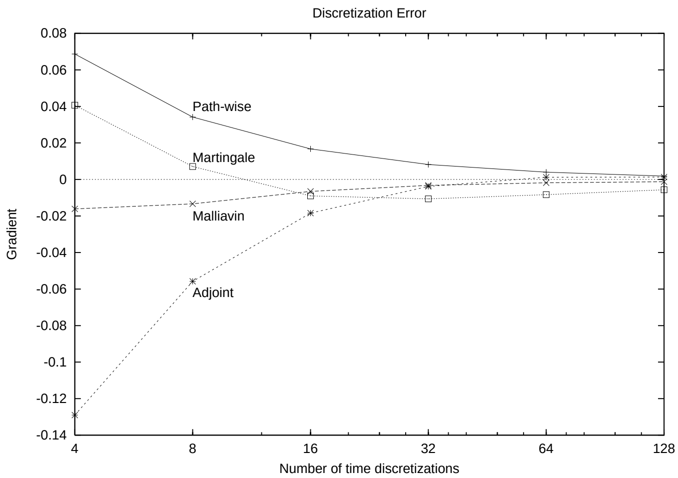


FIGURE 5.2. *Discretization error as a function of the number of time discretizations.*

5.2.2. Discretization error. Here, we report the impact of the number of time discretizations in the regular mesh of the interval $[0,T]$, in the computation of the gradient $\nabla_\alpha J(\alpha)$. Figure 5.2 reports the sensitivity of $J(\alpha)$ (for $\alpha = \alpha^*$) w.r.t. the parameter α_1, computed by different estimators, with $N = 8, 16, 32, 64$ and 128 time discretizations. Recall that, for this setting of the parameter, the true gradient is zero. To get relevant results, we have run 10^7 simulations: this ensures that the width of the confidence interval is less than 10^{-3} for all methods. We can empirically check that the convergence holds at rate $1/N$ (as previously proved), except for the martingale method, for which the rate of convergence is not clear because of the sign change (more discretization times would be needed).

5.3. Sensitivity analysis in a financial market. We consider two risky assets with price process evolving according to the following SDE under the so-called risk neutral probability:

$$\frac{dS_t^1}{S_t^1} = r\,dt + \sigma(S_t^1, \lambda_1)\,dW_t^1$$
$$\frac{dS_t^2}{S_t^2} = r\,dt + \sigma(S_t^2, \lambda_2)\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^2\right)$$

with constant interest rate r and volatility functions $\sigma(x, \lambda) = 0.25(1 + \frac{1}{1+e^{-\lambda x}})$. The parameters of this dynamics are λ_1, λ_2, and the correlation coefficient ρ. Suppose that the true model is given by some set of parameters and that we are interested by the impact of the inaccuracy on these parameters (due to a previous statistical procedure) over option prices. For instance, we may consider digital options with payoff $\chi(S_T^1 - S_T^2)$ (where $\chi(x) = \mathbf{1}_{x\geq 0}$) whose prices are given by $J(\lambda_1, \lambda_2, \rho) = \mathbb{E}[\chi(S_T^1 - S_T^2)]$ up to the discount factor.

TABLE 5.4
Variance of the estimators $H_T^{Mall.Ell.}$, $H_T^{Adj.}$, $H_T^{Mart.}$, $H_T^{\varepsilon,Path.}$.

Var(H) or Var(H^ε)	Malliavin	Adjoint	Martingale	Path-wise $\varepsilon = 10^{-2}$	Path-wise $\varepsilon = 10^{-3}$	Path-wise $\varepsilon = 10^{-4}$
λ_1	0.0011	0.0022	0.0012	0.0053	0.0378	3.8951
λ_2	0.0048	0.0030	0.0018	0.0042	0.0296	4.9427
ρ	1.5788	2.0829	1.4323	1.6523	14.923	100.86
CPU Time	20.8s	18.6s	7.31s	2.97s	2.97s	2.97s

We estimate the sensitivity of J with respect to the parameters λ_1, λ_2, and ρ. Table 5.4 reports the empirical variance of the estimators ($H_T^{Mall.Ell.}$, $H_T^{Adj.}$, and $H_T^{Mart.}$) of the sensitivity of J w.r.t. the parameters for the Malliavin calculus, adjoint and martingale methods. Since the payoff function is not differentiable, we cannot directly apply the path-wise method; instead, we use χ^ε, a $\varepsilon-$regularization of χ defined by $\chi^\varepsilon(x) = 1$ if $x > \varepsilon$, 0 if $x < -\varepsilon$ and $(x+\varepsilon)/(2\varepsilon)$ otherwise. Note that this induces a bias on the true value of gradient, bias which vanishes when ε goes to 0. The path-wise estimator that we obtain with this regularization is denoted by $H^{\varepsilon,Path.}$ and Table 5.4 reports also its associated variance for different values of ε.

For this experiment, we ran 1000 trajectories with initial values $S_0^1 = S_0^2 = 1$, $r = 0.04$, $T = 1$, $h = 0.01$ and parameters setting $\lambda_1 = 2$, $\lambda_2 = 2$, and $\rho = 0.6$.

We notice that the variance obtained by the path-wise methods is significantly larger than those obtained by the other methods (especially when ε is small), which motivates the use of the Malliavin calculus, adjoint or martingale estimators for non-smooth cost functions. For piece-wise smooth cost functions, we could also combine two methods as suggested in [FLL+99] to reduce further the variance: the path-wise method where the cost function is smooth and one of the other ones where it is not.

5.4. Neuro-control for a stochastic target problem. We consider a two-dimensional stochastic target (for example that models the displacement of a fly) moving according to a Brownian motion. We control a squared fly-swatter with a 2-dimensional bounded force $(b(u_1), b(u_2))$ (where $u = (u_1, u_2)$ is the control), and our goal is to hit the fly at time T. Let $X = (X_1, X_2)$ be the relative coordinates of the fly with respect to the fly-swatter, and $V = (V_1, V_2)$ be the velocity of the fly-swatter. A simple model of the dynamics is

$$\begin{aligned} dX_{1,t} &= V_{1,t}dt + \sigma_{fly}dW_t^1 \\ dX_{2,t} &= V_{2,t}dt + \sigma_{fly}dW_t^2 \\ dV_{1,t} &= b(u_{1,t})dt + \sigma_{swat}(1 + ||u_t||)dW_t^3 \\ dV_{2,t} &= b(u_{2,t})dt + \sigma_{swat}(1 + ||u_t||)dW_t^4 \end{aligned}$$

where $b(x) = [1 - e^{-x}]/[1 + e^{-x}]$. The coefficient $(1 + ||u||)$ (where $||u|| = \sqrt{u_1^2 + u_2^2}$) adds uncertainty on highly forced movements. The goal is to reach the fly with the fly-swatter at time T: hence, $J(u(\cdot); X_0, V_0) = \mathbb{E}[\mathbf{1}_{(X_{1,T}, X_{2,T}) \in A}]$ where $A = [-a, a] \times [-a, a]$ is the squared fly-swatter.

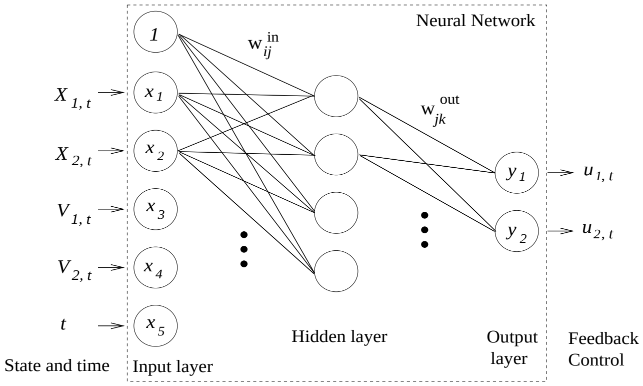


FIGURE 5.3. *The architecture of the network*

We implement the feedback controller using a one-hidden-layer neural network (see [Hay94] or [RM86] for general references on neural networks) whose architecture is given in Figure 5.3. The input layer $(x_i)_{1\leq 5}$ is connected to the state and time variables: $x_1 = X_{1,t}$, $x_2 = X_{2,t}$, $x_3 = V_{1,t}$, $x_4 = V_{2,t}$, and $x_5 = t$. There is one hidden layer with n neurons, and the output layer $(y_k)_{1\leq k\leq 2}$ returns the feedback control $y_1 = u_1(t)$, $y_2 = u_2(t)$. The network is defined by 2 matrices of weights (the parameters): the *input weights* $\{w_{ij}^{in}\}$ and the *output weights* $\{w_{jk}^{out}\}$. The output of the network is given by $y_k = \sum_{j=1}^n w_{jk}^{out}\sigma\left(\sum_{i=0}^5 w_{ij}^{in}x_i\right)$ (for $1 \leq k \leq 2$), where the w_{0j}^{in}'s are the *bias weights* (and we set $x_0 = 1$) and $\sigma(s) = 1/(1 + e^{-s})$ is the sigmoïd function.

In this experiment, we have used a hidden layer with 4 neurons (thus there are $6 \times 4 + 4 \times 2 = 32$ control parameters); we have run 1000 trajectories with initial values of the weights chosen randomly within the range $[-0.1, 0.1]$. Here, $T = 1$, $h = 0.05$, $\sigma_{fly} = \sigma_{swat} = 0.1$ and $a = 0.1$. Each trajectory is started from a initial state chosen randomly within the range $\Omega = [-0.5, 0.5]^4$. Thus, we actually compute $\nabla_w \mathbb{E}[J(\cdot; X_0, V_0) \mid (X_0, V_0) \sim \frac{1}{|\Omega|}\mathbf{1}_\Omega(d\omega)]$, for each weight w.

TABLE 5.5

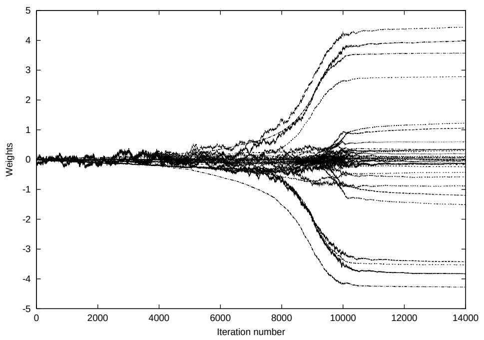
Variance of the estimators of the gradient of $\mathbb{E}[J(\cdot; X_0, V_0)|(X_0, V_0) \sim \frac{1}{|\Omega|}\mathbf{1}_\Omega(d\omega)]$ w.r.t. the weights. The values provided are the averaged variances over all 32 parameters.

Var(H) or Var($H_\varepsilon^{Path.}$)	Malliavin	Adjoint	Martingale	Path-wise $\varepsilon = 10^{-3}$	Path-wise $\varepsilon = 10^{-4}$
Average over all parameters	0.1917	0.2550	0.1701	3.364	187.48
CPU Time	$70.44s$	$22.04s$	$5.73s$	$2.88s$	$2.88s$

Table 5.5 reports the empirical variance of the estimators ($H^{Mall.Ell.}$, $H^{Adj.}$, and $H^{Mart.}$) of the gradient of J w.r.t. the parameters (the set of input and output weights). Here again, the function to be maximized is not differentiable and to apply the path-wise method, we use a $\varepsilon-$regularization of the indicator function of A (i.e. $J^\varepsilon(\alpha) = \mathbb{E}[(\chi^\varepsilon(X_{1,T}+a)-\chi^\varepsilon(X_{1,T}-a))(\chi^\varepsilon(X_{2,T}+a)-\chi^\varepsilon(X_{2,T}-a))])$. The associated path-wise estimator is denoted $H_T^{\varepsilon,Path.}$: its variance for some values of ε is also given in Table 5.5. The resulting large variance makes the path-wise approach inappropriate in this situation, even if the computational time is the lowest one. Besides, the martingale method is the most attractive.

Stochastic Approximation of an optimal controller. We run a stochastic approximation algorithm (5.1) with a learning rate $\eta_k = \frac{K}{K+k}$ (with $K = 1000$) using a neural network with 4 hidden neurons. At each iteration, the SA algorithm uses an estimator of the gradient of J w.r.t. the weights which averages 50 samples of the martingale estimator.

On Figures 5.4, we plot the parameter and performance evolutions w.r.t. the iteration number: we obtain a set of weights that provides a locally optimal performance, but here, there is no guarantee of global optimality of the controller.



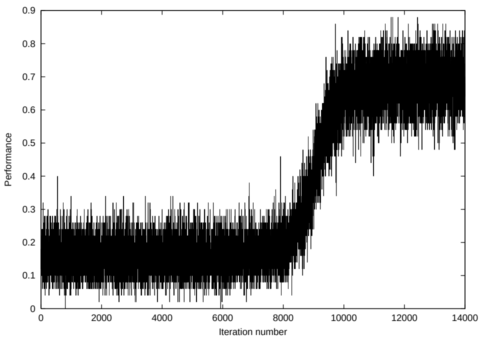


FIGURE 5.4. *Stochastic approximation of the parameters (the weights of the neural network) and performance of the parameterized controller. The gradient is estimated using the martingale method.*

This stochastic gradient algorithm in the space of parameterized policies is often denoted as *policy search* (for which an abundant literature exists in the discrete-time case, see e.g. [BB01]), in contrast to *value search* for which some approximate dynamic programming algorithm is performed on a parameterized value function (see e.g. [BT96b]). One may also combine these approaches and learn an approximate value function to perform a policy search (the so-called *Actor-Critic* algorithms, see e.g. [KB99]).

6. Conclusion. In this work, we have derived three new type formulae to compute $\nabla_\alpha \mathbb{E}(f(X_T^\alpha))$ or $\nabla_\alpha \mathbb{E}(\int_0^T g(t, X_t^\alpha)dt + f(X_T^\alpha))$ by Monte Carlo methods. Our computations rely on Itô-Malliavin calculus and martingale techniques: the representations derived are simple to implement using Euler type schemes and the associate weak error is most of cases linear w.r.t. the time step. We have assumed that f is bounded, but all results remain valid if it satisfies some polynomial growth.

The numerical experiments enable to set the following conclusions on the use of one or an other method.

1) *Path-wise approach.* It can be used only if the instantaneous and terminal costs are differentiable. Otherwise, some regularization procedures lead to high variances. It gives the smallest computational time. Note also that no condition on the non degeneracy of the diffusion coefficient is required. For the implementation, only the first derivatives of the coefficient are needed.
2) *Malliavin calculus approach.* It handles the case of non smooth costs but the computational time is quite large. A non degeneracy assumption has to be satisfied but it may be not stringent (hypoellipticity e.g.). Note that the simulation procedures require the computations of the second derivatives of the coefficients w.r.t. x, x and x, α.
3) *Adjoint approach.* It can be applied in the elliptic case and becomes efficient (for the computational time) when the number of parameters is large. However, it is quite slow, especially when there are instantaneous costs (because of double time integrals and a possible large number of discretization times). The second derivatives required for the simulations concern the ones w.r.t. x, x.
4) *Martingale approach.* The diffusion coefficient has to be elliptic. As for the adjoint approach, it handles situations with non-smooth costs. It turns out to be very fast (almost as fast as the path-wise approach), but it is slower for instantaneous costs problems (same reason than for the adjoint approach). Remark also that only the first derivatives of the coefficient are needed.

In further researches, we will be interested in the analysis of the weak error for the martingale method and performing numerical optimizations in the case of the general non degenerate case (such as portfolio optimization problems in finance).

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